Computation of Magnetization Normal Oscillation Modes in Complex Micromagnetic Systems

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Abstract: The small oscillation modes in complex micromagnetic systems around an equilibrium are analyzed in the frequency domain by using a formulation which naturally preserves the main physical properties of the problem. The Landau-Lifshitz-Gilbert (LLG) equation is linearized around a stable equilibrium configuration. The linear equation is recast as a generalized eigenvalue problem for suitable self-adjoint operators connected to the micromagnetic effective field. The spectral properties of the eigenvalue problem are studied in the lossless limit and in the presence of small dissipative effects. It is shown that the discrete approximation of the eigenvalue problem obtained either by finite difference or finite element methods has a structure which preserves relevant properties of the continuum formulation. Finally, the normal oscillation modes and frequencies are numerically computed for different micromagnetic systems in order to show the generality of the approach.

Keywords: Ferromagnetic resonance, normal modes, resonant frequencies, generalized eigenvalue problem.

1. INTRODUCTION

The determination of the resonances for ferromagnetic nanoparticles is very important to understand the the magnetization dynamics driven by radio-frequency external magnetic fields. The main experimental observations concerning this problem are traditionally related to ferromagnetic resonance phenomena and spin-wave instability (see, for instance, Suhl (1956, 1957); Wigen (1994); Bertotti (2001); d’Aquino (2006)). In this situation, a small ferromagnetic particle is initially saturated by applying a sufficiently strong external DC magnetic field which along a given direction. Afterwards, the particle is illuminated with a radio-frequency (RF) external magnetic field. The amplitude of this RF field is much smaller than the DC component. Then, the frequency of the RF field is in the range of microwaves, typically much larger than the characteristic dimension of the nanoparticle. Therefore, the frequency of the RF field is slowly varied and the power absorbed by the magnetic particle is measured as a function of the applied field frequency. The peaks in this response curve reveal the excitation of certain magnetization normal oscillations.

Another typical situation where the normal oscillations around an equilibrium play a fundamental role, is in the modelling of thermal fluctuations. In fact, thermal agitation tends to slightly perturb the equilibrium magnetization and therefore, from the analysis of the resonant response of the micromagnetic system, one can retrieve insightful information about fluctuation and dissipation processes as it is suggested by experimental observations by Perzlmeier (2005); Bolte (2006).

The theoretical description of magnetization resonance phenomena has been the focus of considerable research. The case of ellipsoidal particles with negligible exchange interaction has been analyzed by Walker (1957). On the other hand, the dual case of spheres in which the exchange interaction was prevalent with respect to magnetostatics has been tackled by Aharoni (1991). The case of an infinite cylinder was treated by Brown (1963). These fundamental works produced analytical techniques to determine the resonant frequencies and modes, but their use was limited to the case of particles with very special shapes. Recently, there has been growing interest in the computation of resonances for particles with generic shapes such as, for instance, rectangular thin-films or ferromagnetic prisms. To this end, numerical techniques based on micromagnetic simulations and Fourier analysis have been proposed, for instance, by McMichael (2005); Grimsditch (2004). In these works, magnetization dynamics around the equilibrium is excited with suitable field pulse and from the Fourier analysis of the response the normal excitations are extracted. These methods have some disadvantages. First of all, when the system is excited by a field pulse, it is very difficult to distinguish the normal oscillations by analyzing the magnetization response. In addition, in the case of low resonant frequencies, the micromagnetic simulations should take very long time in order to have a good time resolution of the oscillation.

In this paper, we propose a general formulation of the problem for arbitrarily shaped particles in terms of eigenvalue problem for suitable linear self-adjoint operators. This approach naturally leads to straightforward numerical computations of magnetization normal resonant modes and frequencies when a spatial discretization is introduced.
The discretization can be based either on finite difference or finite element methods.

2. PROBLEM FORMULATION

Magnetization dynamics is described by the Landau-Lifshitz-Gilbert equation (in normalized form):
\[
\frac{\partial \mathbf{m}}{\partial t} = -\mathbf{m} \times \left( \mathbf{h}_{\text{eff}} - \alpha \frac{\partial \mathbf{m}}{\partial t} \right), \quad \text{in } \Omega
\]
where \(\Omega\) is the region occupied by the magnetic body, \(\mathbf{m} = M/M_s\) is the magnetization vector, \(\mathbf{h}_{\text{eff}} = \mathbf{H}_{\text{eff}}/M_s\) is the effective field, \(M_s\) is the saturation magnetization of the material, \(\alpha\) is the damping constant and time is measured in unit of \((\gamma M_s)^{-1}\) \((\gamma\) is the absolute value of the gyromagnetic ratio). Equation (1) must be complemented with the boundary condition
\[
\frac{\partial \mathbf{m}}{\partial n} = 0 \quad \text{on } \partial \Omega .
\]

The effective field takes into account exchange, magnetostatic, uniaxial anisotropy and Zeeman interactions, and is given by:
\[
\mathbf{h}_{\text{eff}}[\mathbf{m}(\cdot)] = I_{\text{ex}}^2 \nabla^2 \mathbf{m} + \mathbf{h}_m + \kappa_{an} (\mathbf{e}_{an} \cdot \mathbf{m}) \mathbf{e}_{an} + \mathbf{h}_a ,
\]
where \(I_{\text{ex}} = \sqrt{2A/(\mu_0 M_s^2)}\) is the exchange length, \(A\) is the exchange constant, \(\mathbf{h}_m\) is the magnetostatic field, \(\kappa_{an}\) and \(\mathbf{e}_{an}\) are the uniaxial anisotropy constant and unit-vector, respectively, and \(\mathbf{h}_a\) is the external applied field.

2.1 Conservative linear dynamics

Let us assume that the particle magnetization configuration \(\mathbf{m}_0(\mathbf{r})\) is a stable micromagnetic equilibrium, not necessarily spatially uniform, and consider, for the time being, the conservative dynamics, that is the case in which \(\alpha = 0\) in Eq. (1). The small oscillations \(\Delta \mathbf{m}(\mathbf{r}, t)\) of the magnetization \(\mathbf{m}(\mathbf{r}, t) = \mathbf{m}_0(\mathbf{r}) + \Delta \mathbf{m}(\mathbf{r}, t)\) around the equilibrium state can be studied by linearizing Eq. (1) (with \(\alpha = 0\)) around \(\mathbf{m}_0\). The result of this procedure is:
\[
\frac{\partial \Delta \mathbf{m}}{\partial t} = -\mathbf{m}_0 \times (\Delta \mathbf{h}_{\text{eff}} - h_0(\mathbf{r}) \Delta \mathbf{m}) ,
\]
where \(h_0(\mathbf{r}) = \mathbf{m}_0(\mathbf{r}) \cdot \mathbf{h}_{\text{eff}}[\mathbf{m}_0(\mathbf{r})]\) and \(\Delta \mathbf{h}_{\text{eff}}\) is given by:
\[
\Delta \mathbf{h}_{\text{eff}}[\mathbf{m}(\cdot), \mathbf{t}] = I_{\text{ex}}^2 \nabla^2 \Delta \mathbf{m} + \mathbf{h}_m + \kappa_{an} (\mathbf{e}_{an} \cdot \Delta \mathbf{m}) \mathbf{e}_{an} ,
\]
where \(\Delta \mathbf{h}_{\text{mef}}\) is the magnetostatic field produced by the vector field \(\Delta \mathbf{m}\). Equation (4) must be complemented with the boundary condition
\[
\frac{\partial \Delta \mathbf{m}}{\partial n} = 0 \quad \text{on } \partial \Omega .
\]

We immediately observe that, due to the magnetization magnitude conservation property \(|\mathbf{m}| = 1\), one has, at first order in \(\Delta \mathbf{m}\):
\[
|\mathbf{m}_0 + \Delta \mathbf{m}| = 1 \Rightarrow \mathbf{m}_0 \cdot \Delta \mathbf{m} = 0 .
\]

From the latter equation, it can be inferred that the generic unknown perturbation field \(\Delta \mathbf{m}(\mathbf{r}, t)\) has nonzero components only in the plane pointwise perpendicular to the equilibrium magnetization \(\mathbf{m}_0\). This property will have great importance in the sequel when the aspects related to the numerical computations will be addressed. In this respect, we introduce the local frame of unit-vectors \(\mathbf{e}_1(\mathbf{r}), \mathbf{e}_2(\mathbf{r}), \mathbf{e}_3(\mathbf{r})\) defined such that:
\[
\mathbf{e}_1 = -\mathbf{m}_0 \times (\mathbf{e}_2 \times \mathbf{m}_0) , \quad \mathbf{e}_2 = \mathbf{e}_2 \times \mathbf{m}_0 , \quad \mathbf{e}_3 = \mathbf{m}_0 .
\]
operator, it can be shown that the operator $A_0$ is self-adjoint with respect to the standard inner product of $L^2(\Omega)$, defined by
\[
(\tilde{v}, \tilde{w}) = \frac{1}{V_\Omega} \int_\Omega \tilde{v}^* \tilde{w} \, dV .
\] (18)
In the latter equation the notation $^*$ indicates the complex conjugate. The operator $A_0$ is also positive definite, provided that the equilibrium configuration is a stable equilibrium. For instance, this happens when the DC applied field saturates the ferromagnetic particle and is greater than its nucleation field (see Brown (1963)).

It is easy to prove that, if we consider the subspace $H^1_0(\Omega)$ of $H^1(\Omega)$ vector fields having zero component along $e_3$, then also $A_{0\perp} = P_1 A_0$ is self-adjoint and positive definite in $H^1_0(\Omega)$ with respect to the inner product (18), namely
\[
(\tilde{v}, A_{0\perp} \tilde{w}) = (A_{0\perp} \tilde{v}, \tilde{w}) , \quad (\tilde{v}, A_{0\perp} \tilde{w}) > 0 \quad \forall \tilde{v} \neq 0 .
\] (19)
In addition, we observe that the operator $B_0$ is also self-adjoint and, when restricted to the subspace $H^1_0(\Omega)$, is also invertible and coincident with its inverse, namely:
\[
(\tilde{v}, B_0 \tilde{w}) = (B_0 \tilde{v}, \tilde{w}) , \quad B_0 B_0 = I .
\] (20)
As a consequence of these considerations, we can restrict Eq. (15) to the subspace $H^1_0(\Omega)$, as suggested by Eq. (7). One can be easily convinced that the problem of finding the resonances of the particles is equivalent to the following generalized eigenvalue problem (see Weinberger (1974)) for the operators $A_{0\perp}$ and $B_0$:
\[
A_{0\perp} \tilde{v} = \omega B_0 \tilde{v} .
\] (21)
The generalized eigenvalues $\omega_k$ are the resonant frequencies of the micromagnetic system and the generalized eigenfunctions $\tilde{w}_k(r)$ represent the (normal) resonant oscillation modes around the equilibrium configuration.

2.2 Spectral properties of the generalized eigenvalue problem

The eigenvalue problem (21) has considerable properties due to the nature of the involved operators $A_{0\perp}$ and $B_0$. First of all, the generalized eigenvalues are all real. Then, considering two different and non degenerate eigenvalues, the corresponding eigenfunctions are orthogonal. In order to prove these properties let us consider two generic eigenfunctions $\tilde{w}_k$ and $\tilde{w}_h$ such that
\[
A_{0\perp} \tilde{w}_k = \omega_k B_0 \tilde{w}_k ,
\] (22)
\[
A_{0\perp} \tilde{w}_h = \omega_h B_0 \tilde{w}_h .
\] (23)
Now, let us scalar multiply both sides of Eq. (22) and Eq. (23) by $\tilde{w}_h$ and $\tilde{w}_k$ respectively and take the complex conjugate of Eq. (23). One obtains:
\[
(\tilde{w}_h, A_{0\perp} \tilde{w}_k) = \omega_k (\tilde{w}_h, B_0 \tilde{w}_k) ,
\] (24)
\[
(\tilde{w}_h, A_{0\perp} \tilde{w}_h)^* = \omega_h^* (\tilde{w}_h, B_0 \tilde{w}_h)^* .
\] (25)
By subtracting Eq. (25) from Eq. (24), using the property of the scalar product $(v,w) = (w,v)^*$ and the fact that both the operators $A_{0\perp}$ and $B_0$ are self-adjoint, one has after some algebra:
\[
\frac{\omega_h^* - \omega_h}{\omega_k^* - \omega_k} (\tilde{w}_h, A_{0\perp} \tilde{w}_k) = 0 .
\] (26)
It can be clearly seen that, if $k = h$, then, due to the positive definiteness of the operator $A_{0\perp}$ the imaginary part of Im$\omega_k = (\omega_k - \omega_k^*)/2 = 0$. Conversely, when $k \neq h$ and the eigenvalues are distinct, it happens necessarily that:
\[
(\tilde{w}_h, A_{0\perp} \tilde{w}_k) = 0 .
\] (27)
The latter equation expresses the orthogonality condition between different oscillation modes with respect to the new scalar product
\[
(\tilde{v}, \tilde{w})_{A_{0\perp}} = (\tilde{v}, A_{0\perp} \tilde{w}) = (A_{0\perp} \tilde{v}, \tilde{w}) ,
\] (28)
which can be consistently defined since the operator $A_{0\perp}$ is self-adjoint and positive definite.

2.3 Treatment of small dissipation

In the following, we study the perturbation of the eigenfrequencies $\omega_k$ and eigenfunctions $\tilde{w}_k$ which arise from the introduction of a small dissipation in the system, which corresponds to consider the linearized LLG equation in the frequency domain with $\alpha \neq 0$:
\[
-j \omega \Lambda (m_0) \cdot \tilde{v} - j \omega \alpha \tilde{v} = A_{0\perp} \tilde{v} .
\] (29)
By using the operator notations introduced before, one can write Eq. (29) again in the form of generalized eigenvalue problem:
\[
A_{0\perp} \tilde{v} = \omega (B_0 + \delta B) \tilde{v} ,
\] (30)
where the operator $\delta B$ is
\[
\delta B = -j \alpha I .
\] (31)
It can be immediately inferred that the introduction of the damping affects the self-adjointness of the operator $B_0 + \delta B$. Therefore, one consequence is that the natural frequencies $\omega'_k$ given by the eigenvalues of the problem (30), such that,
\[
A_{0\perp} \tilde{w}_k = \omega'_k (B_0 + \delta B) \tilde{w}_k ,
\] (32)
will be, in general, complex and will not coincide with the $\omega_k$ derived in the conservative case. Moreover, the eigenfunctions of the problem (30) will not be orthogonal anymore.

Nevertheless, for small values of the damping constant $\alpha$, which is the case in many experimental situations (typically $\alpha \sim 10^{-3} \div 10^{-2}$), we can make a perturbative study, according to Kato (1980), in order to derive the expressions of $\omega'_k$.

We start writing the eigenvalue problem (32) in perturbative form. In fact, we assume that
\[
\omega'_k = \omega_k + \delta \omega_k , \quad \tilde{w}_k = \tilde{w}_k + \delta \tilde{w}_k , \quad A_{0\perp} \tilde{w}_k = \omega_k B_0 \tilde{w}_k .
\] (33)
With these assumptions, the eigenvalue problem (32) becomes:
\[
A_{0\perp} (\tilde{w}_k + \delta \tilde{w}_k) = (\omega_k + \delta \omega_k) (B_0 + \delta B) (\tilde{w}_k + \delta \tilde{w}_k) .
\] (34)
By expanding the latter equation, remembering (33) and retaining only terms up to first order, one has:
\[
A_{0\perp} \delta \tilde{w}_k = \omega_k B_0 \delta \tilde{w}_k + \omega_k \delta B \tilde{w}_k + \delta \omega_k B_0 \tilde{w}_k .
\] (35)
It is convenient to represent the eigenfunctions perturbations $\delta \tilde{w}_k$ by using the unperturbed eigenfunctions $\tilde{w}_k$:
\[
\delta \tilde{w}_k = \sum_k \epsilon_{kh} \tilde{w}_h ,
\] (36)
where the expansion coefficients $\epsilon_{kh}$ are the Fourier coefficients given by
\[
\epsilon_{kh} = (\delta \tilde{w}_k, \tilde{w}_h)_{A_{0\perp}} .
\] (37)
By using the expansion (36) in Eq. (35), one has:

$$A_{0,l} \sum_{h} \varepsilon_{kh} \hat{w}_h = \omega_k B_0 \sum_{h} \varepsilon_{kh} \hat{w}_h + \omega_k \delta B \hat{w}_k + \delta \omega_k B_0 \hat{w}_k.$$  

(38)

Now, by scalar multiplying both sides by the eigenfunction $\hat{w}_k$, remembering the orthogonality condition (27) and the expression (31) of the operator $\delta \hat{B}$, assuming orthonormal eigenfunctions and solving for the eigenfrequency perturbation $\delta \omega_k$, one ends up with:

$$\delta \omega_k = -\omega_k \left( \frac{(\hat{w}_k, \delta B \hat{w}_k)}{(\hat{w}_k, B_0 \hat{w}_k)} \right) = j \alpha \omega_k^2 \left( \frac{(\hat{w}_k, \hat{w}_k)}{(\hat{w}_k, A_{0,l} \hat{w}_k)} \right)$$

$$= j \alpha \omega_k^2 \left( \frac{(\hat{w}_k, \hat{w}_k)}{(\hat{w}_k, A_{0,l} \hat{w}_k)} \right),$$

(39)

where $\gamma_k = (\alpha \omega_k^2 (\hat{w}_k, \hat{w}_k))^{-1}$. As expected, we notice that, as a consequence of the stability of the equilibrium $\mathbf{m}_0(\mathbf{r})$, the imaginary parts $\delta \omega_k$ of the natural frequencies $\omega_k$ are positive. From the physical point of view, this implies that the magnetization mode $\hat{\mathbf{m}}_l$ will correspond to a damped oscillation towards the equilibrium configuration with the decay constant $\gamma_k$. Moreover, with some algebraic manipulations, the Fourier coefficients of the eigenfunctions perturbations $\delta \hat{w}_k$ can be derived:

$$\epsilon_{kh} = j \alpha \omega_k \left( \frac{(\hat{w}_k, \hat{w}_k)}{(\hat{w}_k, \hat{w}_k)} \right), \quad k \neq h,$$

(40)

$$\epsilon_{kh} = \frac{\alpha}{2} \left( \frac{(\hat{w}_k, \hat{w}_k)}{(\hat{w}_k, \hat{w}_k)} \right), \quad k = h.$$  

(41)

We observe that $\delta \omega_k$, as well as $\epsilon_{kh}$, are small quantities if $\alpha$ is small (typically $\alpha \sim 10^{-3} \div 10^{-2}$) and $\omega_k$ is not close to $\omega_h$. Therefore, the coupling between normal modes arising from the introduction of damping is a relatively small effect.

### 2.4 Spatial discretization of the eigenvalue problem

In the above derivation no assumption has been made on the particle shape. The approach can be used for particles with arbitrary shape and is very interesting as far as the numerical computation of the resonant modes is concerned. In fact, let us suppose to perform a spatial discretization of Eq. (21), either based on finite elements or finite differences, on $N$ nodes or cells. We observe that, since the constraint (7) implies that the normal modes have only nonzero components in the plane locally transverse to $\mathbf{m}_0$, the dimension of the problem is $2N \times 2N$ instead of $3N \times 3N$. This fact allows one to have considerable saving in terms of memory and computational cost. In addition, as a consequence of the self-adjointness and positive definiteness of the operator $A_{0,l}$ in the continuum problem, the discretized version of the operator $A_{0,l}$ will be the symmetric and positive definite matrix $A_{0,l}$. It is important to underline that the matrix $A_{0,l}$ can be assembled by using the same exchange, magnetostatic and anisotropy operators implemented in micromagnetic codes, as it can be seen from Eq. (16). Moreover, since the operator $B_0$ is hermitian, then its discretized version will be the hermitian matrix $B_0$. The discretized eigenvalue problem will be:

$$A_{0,l} \cdot \hat{\mathbf{w}} = \omega B_0 \cdot \hat{\mathbf{w}},$$

(42)

where $\hat{\mathbf{w}} \in \mathbb{C}^{3N}$ denotes the mesh vector containing the magnetization phasors $\hat{\mathbf{m}}_l \in \mathbb{C}^3$ associated with the computational grid points $1, \ldots, N$. In order to reduce the dimension of this problem, let us introduce in the generic $l$-th grid point an orthogonal reference frame whose unit-vectors are given by

$$\mathbf{e}_{1,l} = -\mathbf{m}_{0,l} \times (\mathbf{e}_x \times \mathbf{m}_{0,l}), \quad \mathbf{e}_{2,l} = \mathbf{e}_x \times \mathbf{m}_{0,l}, \quad \mathbf{e}_{3,l} = \mathbf{m}_{0,l}.$$  

(43)

It is evident, for what we outlined in previous sections, that each vector $\mathbf{w}_l$, represented with respect to this basis, will have the third component equal to zero. The coordinate transformation from the cartesian reference frame and the one defined above is represented for each grid point by the orthonormal $3 \times 3$ matrix $R_l$ containing the following direction cosines

$$R_l = \begin{pmatrix} e_{1,l} & e_{2,l} & e_{3,l} \\ e_{1,l} & e_{2,l} & e_{3,l} \\ e_{1,l} & e_{2,l} & e_{3,l} \end{pmatrix}$$

(44)

such that

$$\mathbf{v}_l = R_l \cdot \mathbf{w}_l,$$

(45)

where $\mathbf{v}_l$ contains the components of the vector associated with the $l$-th grid point with respect to the basis $\{e_{1,l}, e_{2,l}, e_{3,l}\}$. As mentioned before, it happens $\mathbf{e}_{3,l} = 0$. The transformation can be expressed also for the mesh vectors by using the $3 \times 3$ block-diagonal matrix $R = \text{diag}(R_1, \ldots, R_N)$:

$$\mathbf{v} = R \cdot \mathbf{w}.$$  

(46)

By using the latter equation and by left-multiplying both sides of the problem (42) by $R_l$, one obtains:

$$A_{0,l} \cdot \mathbf{v}_l = \omega B_0 \cdot \mathbf{v}_l,$$

(47)

where the matrices $A_{0,l}$ and $B_0$ are:

$$A_{0,l} = R A_{0,l} R^T, \quad B_0 = R B_0 R^T.$$  

(48)

One can be easily convinced that $B_0'$ is the $3N \times 3N$ block-diagonal matrix obtained repeating $N$ times the elementary $3 \times 3$ block

$$B_{3 \times 3} = \begin{pmatrix} 0 & j & 0 \\ -j & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$  

(50)

The generalized eigenvalue problem (47) has the same eigenvalues $\omega_k$ as (42), but different eigenvectors $\mathbf{v}_l$. Nevertheless, the eigenvectors $\mathbf{w}_l$ of the original problem are simply given by

$$\mathbf{w}_l = R^T \cdot \mathbf{v}_l.$$  

(51)

The above coordinate transformation allows one to reduce the size of the problem to $2N \times 2N$ by simply considering the generalized eigenvalue problem for the reduced matrices $A_{0,l}'$ and $B_0'$ obtained removing from $A_{0,l}$ and $B_0$ the rows and columns associated with the third cartesian component of each cell vector.

It is worth noting that such discrete eigenvalue problem can be solved by using well-established techniques of numerical linear algebra. In fact, despite the fact that a generalized eigenvalue problem requires, in general, special solution techniques, in our case it can be transformed into a standard eigenvalue problem with low computational cost. Since the matrix $B_0'$ is hermitian, invertible and coincident with its inverse, one ends up with the standard eigenvalue problem

$$D_{0,l} \cdot \mathbf{w} = \omega \mathbf{w},$$

(52)

where the matrix $D_{0,l} = B_0'^{-1} A_{0,l}'$. We observe that assembling this matrix involves only sparse matrix product.
3. NUMERICAL RESULTS

In order to test the effectiveness of the method, we have computed the normal oscillation modes and the resonant frequencies for three different micromagnetic systems.

3.1 Normal modes of a saturated ferromagnetic thin-film

We consider a magnetic thin-film 100 nm × 100 nm × 3 nm without magneto-crystalline anisotropy, at the equilibrium under the action of a saturating DC field along the z-axis $\mathbf{h}_a = 1.2 \mathbf{e}_z$. The spatial discretization has been performed on a structured mesh consisted of square prism cells of 5 nm × 5 nm × 3 nm. The magnetization has been supposed uniform within each cell (see d’Aquino (2008b)). The discrete exchange operator has been obtained from a 7-point laplacian difference approximation, whereas the magnetostatic operator has been computed by using analytical formulas for prism cells reported by Schabes (1987). We have reported in Fig. 1 the frequency $f_k$ and the computed RMS value

$$W_k(r) = \sqrt{\mathbf{\omega}_k \cdot \mathbf{\omega}_k}$$

(53)

for the first six modes. The snapshots of the vector fields corresponding to the first six modes at time $t = 0$ are also reported in figure 1. One can clearly see that the first resonant mode is the spatially (quasi) uniform mode, as expected from physical considerations. The frequency $f_1 = 8.224$ GHz obtained from the numerical computations is reasonably close to the uniform mode frequency provided by the well-known Kittel formula, which is 8.99 GHz; the small discrepancy is due to the fact that the equilibrium configuration is not perfectly saturated due to the prism geometry which produces a non-uniform magnetostatic field. We observe that the equilibrium configuration $\mathbf{m}_0 = \mathbf{e}_z$ is symmetric with respect to 90 degrees rotations of the system. Concerning the normal modes, two situations may happen (see Fig.1): i) the same symmetry is respected. For example, regarding the modes 1,4 one can clearly see that $W_1, W_4$ are invariant for 90 degrees rotations; ii) the RMS value $W_k(r)$ related to a mode $\mathbf{\omega}_k$, which belongs to a pair of degenerate modes $\{\mathbf{\omega}_k, \mathbf{\omega}_{k'}\}$, can be obtained from the 90 degrees rotation of $W_k(r)$. For instance, in Fig.1 one can clearly see that the modes 2 and 3 are degenerate. Then, the RMS value $W_3(r)$ can be obtained by rotating $W_3(r)$ 90 degrees.

3.2 Normal modes around a vortex equilibrium

We consider a magnetic thin-film 200 nm × 200 nm × 3 nm without magneto-crystalline anisotropy. The non uniform equilibrium configuration $\mathbf{m}_0(r)$ represents a Landau (vortex) remanent state (see Fig. 2). This equilibrium configuration has been obtained with dynamical micromagnetic simulations starting from an artificial Landau state. The LLG equation has been numerically integrated with the implicit midpoint rule scheme described by d’Aquino (2005). A strong damping constant $\alpha = 1$ has been chosen in order to have fast relaxation toward the equilibrium. It has been checked that after 10 ns the effective torque acting on the magnetization is below machine accuracy. Therefore, we have assumed the magnetization state at the end of the simulation as an equilibrium configuration. The free oscillations of the magnetic system around this remanent state have been computed with the proposed technique. The eigenvalue problem has been solved with an iterative implicitly-restarted Arnoldi method reported by Lehoucq (1988) by using a tolerance within double-precision machine accuracy. This method can be applied to compute generalized eigenvalues and eigenvectors incrementally, in ascending or descending order of magnitude. The proposed formulation of the problem, can be therefore applied to analyze systems with moderately large number of unknowns. We have reported in Fig. 2 the frequencies $f_k$ and the computed RMS value $W_k(r)$ of each eigenvector $\mathbf{\omega}_k$ for the first 18 modes. First, we notice that the first mode has a natural frequency which is significantly low compared to higher order modes having frequencies in the GHz regime. As one can see from figure 2, this mode represents a translational motion of the vortex core, which is consistent with experimental observations. Concerning
Fig. 2. Numerically computed resonant modes and frequencies for a magnetic square thin-film 200 nm × 200 nm × 3 nm without magneto-crystalline anisotropy, around the Landau (vortex) configuration (right). The color plot represents the RMS value of the pointwise oscillation amplitude. In the computations \( l_{ex} = 5.71 \text{ nm}, \kappa_{an} = 0 \).

Table 1. Computed resonant frequencies for a system of two prolate ellipsoids \((l_{ex} = 7 \text{ nm})\).

<table>
<thead>
<tr>
<th>( d/a )</th>
<th>( f_1 \text{ GHz} )</th>
<th>( f_2 \text{ GHz} )</th>
<th>( f_3 \text{ GHz} )</th>
<th>( f_4 \text{ GHz} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.0 (T)</td>
<td>12.7</td>
<td>13.1</td>
<td>18.0</td>
<td>18.3</td>
</tr>
<tr>
<td>( \infty ) (T)</td>
<td>11.4</td>
<td>11.4</td>
<td>16.2</td>
<td>16.2</td>
</tr>
<tr>
<td>3.0 (L)</td>
<td>6.12</td>
<td>7.30</td>
<td>15.6</td>
<td>16.3</td>
</tr>
<tr>
<td>( \infty ) (L)</td>
<td>6.02</td>
<td>6.02</td>
<td>15.1</td>
<td>15.1</td>
</tr>
</tbody>
</table>

the normal modes symmetries, the same considerations as in previous section can be applied.

3.3 Normal modes of a system of two prolate ellipsoids

We have applied the finite element method described by Forestiere (2009) to study the oscillation modes in a system with two spheroidal nanoparticles with semiaxes \( a = b = 10 \text{ nm} \) and \( c = 30 \text{ nm} \) (see Fig.3) and distance from the particle centers \( d \). The mesh of each spheroid has 1440 tetrahedral elements and 395 nodes. In Table 3.3, we have reported the first four resonance frequencies related to equilibrium magnetizations orthogonal (T) and parallel (L) to the chain axis. It is apparent from fig. 3 that, in the (T) case, the first and the second modes may be interpreted as two parallel and antiparallel dipoles, respectively, whereas the contrary happens in the (L) case.

4. CONCLUSION

We have proposed a general formulation to determine the resonant frequencies and magnetization normal modes for micromagnetic systems of arbitrary shape in terms of generalized eigenvalue problem for suitable self-adjoint operators. This approach can be used for arbitrary geometry and with any spatial discretization. Numerical computations have verified the effectiveness of the approach.

REFERENCES

M. Bonte et al. (2006), Physical Review B 73, 052406.
R.D. McMichael et al. (2005), J. Appl. Phys. 97, 10J901.
M. d'Aquino et al. (2008), Physica B 403, 242.
C. Forestiere et al. (2009), J. Appl. Phys 105, 07D312.