Regularization of a Disk in a Frictionable Wedge

Julian Newman

Imperial College, London, UK
(e-mail: julian.newman07@imperial.ac.uk)

Abstract: It is known that the motion of bodies in frictional contact with each other is not always determinable within the principles of rigid-body mechanics. One example of such is when forces are applied to a disk lying inside a V-shaped wedge. Here, we approach this example by considering a linear regularization of the wedge. Under certain conditions, we obtain results which, on the one hand, are meaningful independently of any given regularization of the wedge, and yet which, on the other hand, cannot be obtained by consideration of rigidity alone.

Keywords: Mechanical systems; Two-dimensional systems; Dry friction; Regularization; Determinism.

1. THE PROBLEM

Consider, in two dimensions, a rigid disk with radially symmetric distribution of mass, lying at rest between the two straight-line walls of a fixed V-shaped frictionable wedge (Figure 1).

Let $R$ be the radius of the disk. Define the vertical axis to be the axis of symmetry of the wedge, with the corner of the wedge being below the center of the disk. Let $\theta$ be the angle between each wall and the horizontal. Let the coefficient of static friction between the disk and the wedge (at both walls) be $\mu$.

The four contact forces (as shown in Figure 1) will be regarded as “internal” forces; any other forces acting on the disk will be called “external” forces. We have not specified what external forces (if any) are acting on the disk, but merely that the disk must be at rest in equilibrium.

While the disk is lying at rest inside the wedge, suppose that we then apply additional external forces to the disk (henceforth called “applied forces”). Let $F_1$ and $F_2$ be respectively the horizontal and vertical projection of the vector total of all the external forces, and let $M$ be the total anticlockwise moment of the all external forces about the center of the disk.

As in Figure 1, let $N_1$ and $N_2$ be the normal contact forces exerted on the disk respectively by the left and right wall. Let $T_1$ and $T_2$ be the frictional forces exerted on the disk respectively by the left and right wall, each measured as positive if it acts in the direction up the wall.

Excluding any possibilities of a reverse-time-chattering motion, the disk can perform one of the following motions:

(a) remaining at rest in equilibrium;
(b) rolling on the spot (sliding against both walls);
(c) rolling up one of the walls in grip;
(d) rolling up one of the walls, sliding upwards;
(e) rolling up one of the walls, sliding downwards;
(f) leaving contact with the wedge altogether.

For as long as the disk remains in equilibrium, we have the following equilibrium equation:
\[
\begin{pmatrix}
-\sin(\theta) & \cos(\theta) & \sin(\theta) & -\cos(\theta) \\
-\cos(\theta) & -\sin(\theta) & -\cos(\theta) & -\sin(\theta) \\
0 & -R & 0 & -R
\end{pmatrix}
\begin{pmatrix}
N_1 \\
T_1 \\
N_2 \\
T_2
\end{pmatrix}
= \begin{pmatrix}
F_1 \\
F_2 \\
M
\end{pmatrix}. \tag{1}
\]

(Note that this is a system of three equations in four unknowns.)

Suppose that \( \mu < \frac{1}{\tan \theta} \). Then, given the values of \( F_1, F_2 \) and \( M \), one can find which of the six motions described above the disk will undergo. Now suppose that \( \mu > \frac{1}{\tan \theta} \). Then, given the values of \( F_1, F_2 \) and \( M \), it is still true that at most one of the motions (b) to (f) is possible; however, one can show that whatever the values of \( F_1, F_2 \) and \( M \) are, the disk can still remain at rest in equilibrium, while adhering to all the principles of rigid-body mechanics.

So, if the coefficient of friction is very high or if the opening angle of the wedge is very small, then for certain values of the external forces it cannot be determined whether the disk will stay still in equilibrium or not.

2. GOALS OF THIS PAPER IN RELATION TO OTHER WORK

Examples of setups involving friction where “existence and uniqueness” of possible motions becomes problematic date back to Jellett (1872) and Painlevé (1895). Some general theory has been formulated for such problems since then, but relatively little work has been done specifically on a disk in contact with two frictionable walls.

McNamara et al. (2005) considered the disk-in-a-wedge problem. As in this paper, they considered a linear regularization of the wedge. They obtained conclusions which relate to both how the contact forces vary over time while \( |T_i| < \mu N_i \) (for \( i = 1 \) and 2) and what happens when \( |T_i| \) reaches \( \mu N_i \) (for \( i = 1 \) or 2). We will actually study only the former. The key difference is that they restricted to the case that the “normal stiffness” (governing the normal contact forces) and the “tangential stiffness” (governing the friction forces while in the sticking state) are equal. There is no reason to expect this to be realistic in practice; they restricted to this particularly simple case because it still serves to make the point that they were seeking to demonstrate—namely, that switching from the rigid case to the regularized case converts indeterminacy into memory-dependent determinism. As they pointed out, their argument for this can be generalized to the case that the two stiffnesses are different from each other. For each set of values for the stiffnesses, the behaviour of the disk is deterministic; the next natural question to ask, in trying to resolve the original indeterminacy, is whether any uniformity can be observed in the behaviour of the disk across varying stiffnesses values. This is what we shall address here.

The motivation behind our investigation here was a set of numerical simulations that were informally carried out by Dr Wolfgang Stamm and Prof Alexander Fidlin (engineers then working for LuK GmbH & Co.), in which the normal and tangential stiffnesses were not restricted to being equal; the results of these simulations were not published, but Dr Stamm informed the Mathematics Department at Imperial College of them. Given fixed initial values for the contact forces, they observed uniform behaviour of the disk across significant variation in the stiffnesses. I am grateful to Dr Stamm and Prof Fidlin for sharing with us these results.

Accordingly, we will not assume that the normal and tangential stiffnesses are equal; nonetheless (in accordance with the engineers’ numerical findings) we will still obtain formulae that make no explicit reference to the ratio between them. This said, we will not work with complete freedom as to what the ratio between the two stiffnesses is. We will need that their ratio is “not too extreme”—in other words, that the ratio of the larger stiffness to the smaller stiffness is not too large compared to the value of the smaller. (The dimensional imbalance here is recompensed by the other physical variables involved.) This may still turn out to be an unrealistic assumption in some cases, but it is, at least, a significant advance.

Two more important points should be made. Firstly, in both studies mentioned above, the only external forces acting on the disk were gravity, acting vertically downward, and an applied torque. We shall be slightly more general: the only restriction that we shall make is that the applied forces must “consist only of a torque and a horizontal force”; in other words, the vector total of the applied forces must have zero vertical projection.

Secondly, while modelling the sticking state, McNamara et al. worked with the assumption that the forces and torques acting on the disk were always approximately in equilibrium. We will also need this approximation, and we will give some quite general analytical conditions in which this approximation is valid.

It should also be mentioned that Waltersberger (2007) considered the disk in a regularized wedge (again under gravity and an applied torque), and specifically studied stability of sliding solutions.

3. REGULARIZED SETUP

As stated, we will seek to resolve the ambiguity by means of contact regularization. In order to gain information from this approach, it will be necessary to make certain assumptions about the wedge and certain restrictions on the applied forces. (In other words, it is unlikely that the ambiguity can be resolved in full generality.)
We will consider the rigid disk in a linearly regularized wedge. This means that the normal contact force at each wall is directly proportional to the “normal deformation” of the wall, which will be some measure the distance through which the disk is pressed into the wall. We will also linearly regularize the frictional force; this means that at each wall, if the disk is in the sticking state, then tangential relative motion between the disk and the wall can still occur, but such motion will contribute to the “tangential deformation” of the wedge—which, in turn, is directly proportional to the frictional force (which seeks to oppose such relative motion).

The above notions are somewhat vague, but in the next sections we will see more explicitly what the essential properties of such a “linear regularization” of the wedge are, as well as a specific example.

We will need to assume that the two walls are “made of the same material as each other”, i.e. have the same regularization as each other. Thus, the wedge will be genuinely symmetrical about the vertical axis. Viewing the normal contact forces and frictional forces as springs, we will also need that the frictional forces are not too much stiffer than the normal contact forces nor vice versa.

We will regard the external forces which act on the disk as functions of time. Let $t=0$ be the “starting time”. We will assume that, for time $t \leq 0$, the external forces acting on the disk are constant (with vector total $(F_1(0), F_2(0))$ and total moment $M(0)$ about the center of the disk) and that the disk is at rest in equilibrium, sticking to both walls. We can think of these external forces as “natural external forces” (e.g. gravity).

Starting from time $t = 0$, we then apply additional “artificial” external forces. We will assume the following:

- $F_2(t)$ remains constant after time 0. In other words, $F_2(t) = F_2(0)$ for all $t \geq 0$;

- for some $T > 0$, $F_1(\cdot)$ and $M(\cdot)$ are continuously differentiable (or just continuous and piecewise continuously differentiable) on $[0,T]$.

The restriction that $F_2$ must remain constant means that the applied forces (represented by the changes in $F_1$ and $M$ since time 0) are “antisymmetric” about the vertical axis—i.e. if you “look at the setup from behind”, then $\Delta F_1$ will become $-\Delta F_1$ and $\Delta M$ will become $-\Delta M$ (where $\Delta$ denotes “change in value since time $t=0$”).

Finally, define the origin to be the position where the center of the disk would be (while the disk is in contact with both walls) if the walls were completely rigid. At each wall, if the disk penetrates the wall, then the distance through which the disk penetrates the wall at its point of maximal penetration is precisely a linear function of the co-ordinates of the center of the disk.

4. ANALYSIS

The disk will remain in the sticking state until the first time that $|T_i|$ would exceed $\mu N_i$ (for $i = 1$ or 2) if we were to assume that the sticking state persisted further. While the disk is in the sticking state at both walls, let us assume (since the regularization is linear) that its motion can be described by a second order linear differential equation. This second order equation can then be turned into a first order linear equation (in a phase space $\mathbb{R}^n$), by introducing velocity variables. So, letting the phase space incorporate both position and velocity variables of the disk (where the tangential deformations of the wedge are included among the position variables of the disk), we will assume that the motion of the disk obeys a differential equation of the form

\[
\begin{cases}
\dot{x}(t) = A(x(t) - x_0) + \Delta F_1(t)c_1 + \Delta M(t)c_2 \\
x(0) = x_0
\end{cases}
\]

(2)

where $A$ is a constant matrix, and $c_1$ and $c_2$ are constant vectors. The term $Ax(t)$ represents the contact forces. The term $-Ax_0$ represents the natural external forces. Define the “artificial external force” $\Delta F : [0,T] \to \mathbb{R}^n$ as

\[
\Delta F(t) = \Delta F_1(t)c_1 + \Delta M(t)c_2.
\]

For any continuous function $G : [0,T] \to \mathbb{R}^n$, define the function $x^G : [0,T] \to \mathbb{R}^n$ as

\[
x^G(t) = x_0 + \int_0^t e^{sA}G(t-s)ds.
\]

So on time $[0,T]$, $x^{\Delta F}$ is the solution of (2). Notice that $x^G - x_0$ depends linearly on $G$.

Now (2) comes, more specifically, from the following assumption: that while the disk is sticking at both walls, each of the four contact forces can be expressed as a linear functional on the phase space $\mathbb{R}^n$. (In fact, this assumption can still be true if we include damping in the regularization of the wedge, provided that this damping is also linear.) We can denote these four linear functionals respectively as $N_1, T_1, N_2, T_2 : \mathbb{R}^n \to \mathbb{R}$.

For any continuous function $G : [0,T] \to \mathbb{R}^n$, define “the change in $N_1$ (since time 0) under $G$”, denoted

\[
\Delta N_1^G : [0,T] \to \mathbb{R},
\]

as

\[
\Delta N_1^G(t) = N_1(x^G(t)) - N_1(x_0).
\]

Define $\Delta T_1^G$, $\Delta N_2^G$ and $\Delta T_2^G$ similarly. It is easy to check that these depend linearly on $G$.

Now we know that the regularized case must approximate the rigid case. Hence we will have that, for all $t \in [0,T]$,

\[
\begin{pmatrix}
-\sin(\theta) & \cos(\theta) & \sin(\theta) & -\cos(\theta) \\
-\cos(\theta) & -\sin(\theta) & -\cos(\theta) & \sin(\theta) \\
0 & R & 0 & -R \\
\end{pmatrix}
\begin{pmatrix}
f_0^t N_1(x^{\Delta F}(s)) ds \\
f_0^t T_1(x^{\Delta F}(s)) ds \\
f_0^t N_2(x^{\Delta F}(s)) ds \\
f_0^t T_2(x^{\Delta F}(s)) ds
\end{pmatrix}
\]

\[
= \frac{\sin(\theta)}{\cos(\theta)}
\begin{pmatrix}
f_0^t N_1(x^{\Delta F}(s)) ds \\
f_0^t T_1(x^{\Delta F}(s)) ds \\
f_0^t N_2(x^{\Delta F}(s)) ds \\
f_0^t T_2(x^{\Delta F}(s)) ds
\end{pmatrix}
\]
The antisymmetry of the applied forces and the symmetry regularization being studied in this paper.

Effectively, (3) says that the velocity and angular velocity of the disk are small. If we are pedantic, approximation to the rigid case only means that the displacement and angular displacement of the disk away from its initial position and orientation are small. This could still allow for high-frequency oscillations whose maximum velocity is not small. However, a similar argument to that used in section 5 will show that the velocity variables of the disk must be small (provided that the applied forces, as functions of time, are bounded and not too pathological).

We would like to be able to deduce from (3) that

\[
\begin{pmatrix}
-\sin(\theta) & \cos(\theta) & \sin(\theta) & -\cos(\theta) \\
-\cos(\theta) & -\sin(\theta) & -\cos(\theta) & -\sin(\theta) \\
0 & R & 0 & -R
\end{pmatrix}
\begin{pmatrix}
N_1(x^\Delta F(t)) \\
T_1(x^\Delta F(t)) \\
N_2(x^\Delta F(t)) \\
T_2(x^\Delta F(t))
\end{pmatrix}
\approx
\begin{pmatrix}
F_1(t) \\
F_2(t) \\
M(t)
\end{pmatrix},
\]

(4)

The problem, though, is that if the contact forces undergo very high frequency oscillations then we may not be able to conclude this. However, our requirements in section 3 included that the applied force terms are continuously differentiable on \([0, T]\), and that the ratio between the two stiffnesses is not too extreme (in either direction). In section 5, we shall demonstrate that, from these assumptions, it is reasonable to expect such high frequency oscillations to have low amplitude, and so we can conclude that (4) holds. We will demonstrate this by analyzing a simple and effective prototype for the kind of regularization being studied in this paper.

If (4) holds, then it follows that

\[
\begin{pmatrix}
-\sin(\theta) & \cos(\theta) & \sin(\theta) & -\cos(\theta) \\
-\cos(\theta) & -\sin(\theta) & -\cos(\theta) & -\sin(\theta) \\
0 & R & 0 & -R
\end{pmatrix}
\begin{pmatrix}
\Delta N_1^\Delta F(t) \\
\Delta T_1^\Delta F(t) \\
\Delta N_2^\Delta F(t) \\
\Delta T_2^\Delta F(t)
\end{pmatrix}
\approx
\begin{pmatrix}
\Delta F_1(t) \\
\Delta F_2(t) \\
\Delta M(t)
\end{pmatrix},
\]

(5)

The linearity of \(\Delta N_{1i}^G\) and \(\Delta T_{1i}^G\) with respect to \(G\) yields that

\[
\begin{cases}
\Delta N_{1i}^F(t) = -\Delta N_{2i}^F(t) \\
\Delta T_{1i}^F(t) = -\Delta T_{2i}^F(t).
\end{cases}
\]

(6)

The antisymmetry of the applied forces and the symmetry of the wedge yield (by a simple symmetry argument) that

\[
\begin{cases}
\Delta N_{1i}^F(t) = \Delta N_{2i}^F(t) \\
\Delta T_{1i}^F(t) = \Delta T_{2i}^F(t).
\end{cases}
\]

(7)

Combining (6) and (7) yields

\[
\begin{cases}
\Delta N_{1i}^\Delta F(t) = -\Delta N_{2i}^\Delta F(t) \\
\Delta T_{1i}^\Delta F(t) = -\Delta T_{2i}^\Delta F(t).
\end{cases}
\]

(8)

In (5), there are three (approximate) equations for the four unknowns \(\Delta N_{1i}^F\), \(\Delta T_{1i}^F\), \(\Delta N_{1i}^F\) and \(\Delta T_{2i}^F\). Now (8) provides a further two equations in these four unknowns. So, combining these gives a system of five equations in the four unknowns. It turns out that one equation is redundant, and we can solve the system. Solving the system then gives us the formulae which have been the goal of this paper:

\[
\begin{cases}
\Delta N_{1i}^\Delta F = -\frac{1}{2\sin(\theta)}\Delta F_1 + \frac{1}{2R\tan(\theta)}\Delta M \\
\Delta N_{2i}^\Delta F = \frac{1}{2\sin(\theta)}\Delta F_1 - \frac{1}{2R\tan(\theta)}\Delta M \\
\Delta T_{1i}^\Delta F = \frac{1}{2R}\Delta M \\
\Delta T_{2i}^\Delta F = \frac{1}{2R}\Delta M.
\end{cases}
\]

(9)

5. OSCILLATIONS IN THE CONTACT FORCES

In this section, we will seek to justify the approximation that, while the disk is in the sticking state, the forces and torques on the disk are in equilibrium—i.e. (4) holds. We will do this by examining the essential prototype for a linear regularization of the sticking state. It is the same regularization that the engineers working for LuK used for their numerical simulations (except that we will exclude damping).

Let \((x, y)\) and \((u, v)\) be respectively the position and velocity of the center of the disk, and let \(\omega\) be the angular velocity of the disk (positive if anticlockwise). Let \(m\) and \(J\) be respectively the mass of the disk and the moment of inertia of the disk about its center. Define the “normal deformation” at each wall to be the distance of maximal penetration in the direction normal to the wall, as shown in Figure 2. The tangential deformation at each wall cannot be determined at any instant in time just from the position of the disk at that instant in time; rather, we will define its time-derivative to be equal to the tangential relative velocity. So, letting \(s_1\) and \(s_2\) be the tangential deformation respectively at the left and the right wall, we define

\[
\begin{align*}
s_1 &= u\cos(\theta) - v\sin(\theta) + R\omega \\
s_2 &= -u\cos(\theta) - v\sin(\theta) - R\omega.
\end{align*}
\]

We will now imagine fixing the ratio between the normal stiffness and the tangential stiffness, and letting these stiffnesses together tend to infinity. Fix positive values \(k_N\) and \(k_T\), and let the normal and tangential stiffnesses be respectively \(k_N/\varepsilon\) and \(k_T/\varepsilon\), where \(\varepsilon\) is a variable that will tend to zero. We will assume no damping.

The phase space has co-ordinates

\[
x = (u, v, \omega, x, y, s_1, s_2) \in \mathbb{R}^7.
\]
Letting $C$ denote a generic contact force, each of the four contact forces can be expressed in the form

$$C(x) = \frac{1}{\varepsilon} n \cdot x$$  \hspace{1cm} (10)$$

for some $n = (0, 0, n_4, n_5, n_6, n_7) \in \mathbb{R}^7$ which does not depend on $\varepsilon$. The matrix $A$ is given by

$$A = \begin{pmatrix} 0 & \frac{1}{\varepsilon} K \\ 1 & 0 \end{pmatrix} \in \mathbb{R}^{7 \times 7}$$

where

$$K = \begin{pmatrix} \frac{-2k_N \sin^2(\theta)}{m} & 0 & -\frac{k_p \cos(\theta)}{m} & \frac{k_p \cos(\theta)}{m} \\ 0 & \frac{-2k_N \cos^2(\theta)}{m} & \frac{k_p \sin(\theta)}{m} & \frac{k_p \sin(\theta)}{m} \\ 0 & 0 & -\frac{k_p R}{m} & \frac{k_p R}{m} \end{pmatrix} \in \mathbb{R}^{3 \times 4}$$

and

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \cos(\theta) & -\sin(\theta) & R \\ -\cos(\theta) & -\sin(\theta) & -R \end{pmatrix} \in \mathbb{R}^{4 \times 3}.$$

Let us denote the standard unit vectors as $e_i$, $i = 1, \ldots, 7$. Then (2) becomes

$$\begin{cases} \ddot{x}(t) = A(x(t) - x_0) + \frac{1}{m} \Delta F_1(t) e_1 + \frac{1}{m} \Delta M(t) e_3 \\ x(0) = x_0. \end{cases}$$  \hspace{1cm} (11)$$

It is easy to check that $A$ has rank 6, meaning that the dimension of its kernel is 1. This is fundamentally linked to the fact that (1) has four unknown forces but consists of only three equations.

We can define the energy functional $E : \mathbb{R}^7 \rightarrow \mathbb{R}$ on the phase space to be the total of the kinetic energy of the disk and the elastic potential energy in the wedge, that is,

$$E(x) = \frac{1}{2} m u^2 + \frac{1}{2} m v^2 + \frac{1}{2} J \omega^2$$

$$+ \frac{k_N}{2\varepsilon} (\sin(\theta)x + \cos(\theta)y)^2$$

$$+ \frac{k_N}{2\varepsilon} (\sin(\theta)x - \cos(\theta)y)^2 + \frac{k_p}{2\varepsilon} x_1^2 + \frac{k_p}{2\varepsilon} x_2^2.$$

It is easy to see that there exist co-ordinates on the phase space in which the energy functional is simply equal to the square of the distance from the origin. In these co-ordinates, all solutions of the differential equation

$$\dot{x} = A x$$  \hspace{1cm} (12)$$

stay on the same 6-sphere about the origin. So, as we would expect given the physical setup (with no damping), all solutions of (12) are both bounded and bounded away from the origin. (In physical terms: the disk perpetually oscillates inside the wedge, due to being pushed back and forth by the regularized walls.) Thus $A$ must have only imaginary eigenvalues and must be diagonalizable.

One can readily verify that there exist $\bar{c}, \bar{d} \in \mathbb{R}^7$, independent of $\varepsilon$, such that

$$\begin{cases} \varepsilon A \bar{c} = \frac{1}{m} e_1 \\ \varepsilon A \bar{d} = \frac{1}{m} e_3. \end{cases}$$

Working with the decomposition of the phase space into the eigenspaces of $A$, let us subtract from $\bar{c}$ and $\bar{d}$ their respective projections onto the $0$-eigenspace (i.e. the kernel), and let $c$ and $d$ be the results. We will still have

$$\begin{cases} \varepsilon A c = \frac{1}{m} e_1 \\ \varepsilon A d = \frac{1}{m} e_3. \end{cases}$$

It is easy to verify that, even though $A$ depends on $\varepsilon$, the kernel of $A$ is independent of $\varepsilon$. Thus $c$ and $d$ are independent of $\varepsilon$.

Letting $x(t)$ be the solution of (11), define the function $y : [0, T] \rightarrow \mathbb{R}^7$ by

$$y(t) = x(t) - x_0 + \varepsilon (\Delta F_1(t) c + \Delta M(t) d).$$

Then $y(t)$ satisfies the differential equation

$$\begin{cases} \dot{y}(t) = A y(t) + \varepsilon \left( (\Delta F_1)'(t) c + (\Delta M)'(t) d \right) \\ y(0) = 0. \end{cases}$$

We will now rescale the last four co-ordinates of the phase space up by a factor of $1/\sqrt{\varepsilon}$: let us define the function $y^* : [0, T] \rightarrow \mathbb{R}^7$ by

$$y^*(t) = \begin{pmatrix} e_1 \cdot y(t) \\ e_2 \cdot y(t) \\ e_3 \cdot y(t) \\ \frac{1}{\sqrt{\varepsilon}} e_4 \cdot y(t) \\ \frac{1}{\sqrt{\varepsilon}} e_5 \cdot y(t) \\ \frac{1}{\sqrt{\varepsilon}} e_6 \cdot y(t) \\ \frac{1}{\sqrt{\varepsilon}} e_7 \cdot y(t) \end{pmatrix}.$$

Then $y^*$ satisfies the differential equation

$$\begin{cases} \dot{y}^*(t) = \frac{1}{\sqrt{\varepsilon}} A^* y^*(t) + \sqrt{\varepsilon} \left( (\Delta F_1)'(t) c + (\Delta M)'(t) d \right) \\ y^*(0) = 0. \end{cases}$$
where

\[ A^* = \begin{pmatrix} 0 & K \\ 1 & 0 \end{pmatrix}. \]

Note that \( A^* \) is independent of \( \varepsilon \). Now let \( P \in \mathbb{C}^{7 \times 7} \) be an invertible matrix whose columns are eigenvectors of \( A^* \) and (without loss of generality) whose last column is a member of the kernel of \( A^* \). It is straightforward to show that \( P^{-1}y^*(t) \) takes the form

\[
\begin{align*}
\frac{1}{\sqrt{\varepsilon}} & \int_0^x e^{\lambda s} \left( (\Delta F_i)'(t-s)c_1 + (\Delta M)'(t-s)d_1 \right) ds \\
\frac{1}{\sqrt{\varepsilon}} & \int_0^x e^{\lambda s} \left( (\Delta F_i)'(t-s)c_2 + (\Delta M)'(t-s)d_2 \right) ds \\
\frac{1}{\sqrt{\varepsilon}} & \int_0^x e^{\lambda s} \left( (\Delta F_i)'(t-s)c_3 + (\Delta M)'(t-s)d_3 \right) ds \\
\sqrt{\varepsilon} & \int_0^x e^{\lambda s} \left( (\Delta F_i)'(t-s)c_4 + (\Delta M)'(t-s)d_4 \right) ds \\
\sqrt{\varepsilon} & \int_0^x e^{\lambda s} \left( (\Delta F_i)'(t-s)c_5 + (\Delta M)'(t-s)d_5 \right) ds \\
\sqrt{\varepsilon} & \int_0^x e^{\lambda s} \left( (\Delta F_i)'(t-s)c_6 + (\Delta M)'(t-s)d_6 \right) ds \\
0 & \int_0^x e^{\lambda s} \left( (\Delta F_i)'(t-s)c_7 + (\Delta M)'(t-s)d_7 \right) ds 
\end{align*}
\]

where everything which appears in the above expression (apart from \( \varepsilon \) itself) is independent of \( \varepsilon \). The values \( \lambda_i \) and \( \lambda_i \) \((i = 1, 2, 3)\) are the non-zero eigenvalues of \( A^* \), and are imaginary.

Since \( (\Delta F_i)' \) and \( (\Delta M)' \) are continuous on \([0, T]\), they are also integrable on \([0, T]\), and so the Riemann-Lebesgue lemma gives us that

\[
\frac{1}{\sqrt{\varepsilon}} P^{-1}y^*(t) \to 0 \quad \text{as} \quad \varepsilon \to 0
\]

for each \( t \in [0, T] \). Furthermore: the interval \([0, T]\) is compact, and (letting \( \delta \) be an arbitrary positive number) the function

\[ g_G : [0, T] \to L^1 \left([0, T + \delta]\right) \]

given by

\[ g_G(t)(s) = \begin{cases} G(t-s) & s \in [0, t] \\ 0 & s \in (t, T + \delta) \end{cases} \]

is continuous for any continuous function \( G : [0, T] \to \mathbb{C} \). Hence, by the uniform version of the Riemann-Lebesgue lemma for compact subsets of \( L^1(T) \) (see, for example, the remark after Theorem 2.8 in Katznelson, 2004), the convergence of \( \frac{1}{\sqrt{\varepsilon}} P^{-1}y^* \) to \( 0 \) is uniform on \([0, T]\).

Thus \( \frac{1}{\sqrt{\varepsilon}} y^* \) converges to \( 0 \) uniformly on \([0, T]\). Hence by (13), we have that

\[
\frac{1}{\sqrt{\varepsilon}} e_i \cdot y \to 0 \quad \text{uniformly on} \quad [0, T] \\
\text{as} \quad \varepsilon \to 0
\]

for \( i = 4, \ldots, 7 \). It follows, by (10), that

\[
C(t) - C(0) \overset{\text{as}}{\to} -(n \cdot c)\Delta F_i(t) - (n \cdot d)\Delta M(t) \quad (14)
\]

uniformly on \([0, T]\).

For each of the four contact forces, one can compute the values of \( n \cdot c \) and \( n \cdot d \), and one obtains that in each case (14) agrees with the formulae given in (9).

So in summary: It is true that, as the disk oscillates inside the wedge, the contact forces undergo high-frequency oscillations about the values predicted by (9). However, the uniform Riemann-Lebesgue lemma gives us that these oscillations are “small”: while fixing the ratio between the normal stiffness and the tangential stiffness, in the limit as the regularized case tends to the rigid case, the amplitudes of these oscillations tend to zero.

Note, however, that problems can arise in the “true limit” as the linearly regularized case tends to the rigid case (that is, when we take a limit without placing any restrictions on the ratio between the two stiffnesses)—namely:

The uniform Riemann-Lebesgue lemma essentially tells us that if the compliance of the wedge is small, then the oscillations in the motion of the disk will be small compared to the compliance of the wedge. However, if one of the two stiffnesses is many orders of magnitude higher than the other, then the “compliance of the wedge” will effectively be represented by the reciprocal of the smaller stiffness. Thus, the higher-stiffness contact forces may still undergo large oscillations.

6. CONCLUSIONS

Suppose we know the values of the contact forces at time \( t = 0 \) (e.g., by measuring them). Then, under certain conditions (as described in this paper), for as long as the disk remains in the sticking state, (9) will tell us how to track variations in the contact forces based on the variations in the forces which we apply. Indeed, the disk will remain in the sticking state at least until the first time that (9) would predict \( |T_i| \) exceeding \( \mu N_i \) for either \( i = 1 \) or \( i = 2 \). (Again, this is under the same conditions). None of this could have been deduced from purely rigid considerations.

REFERENCES


