Stochastic Behavior of Dissipative Hamiltonian Systems with Limit Cycles

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Abstract: In this paper noise analysis of non-linear oscillator circuits are discussed. For this purpose the physical based concept of dissipative Hamiltonian systems should be applied to a certain class of systems with limit cycle behavior.

Keywords: non-linear systems, noise analysis, oscillator circuits, Hamilton mechanics.

1. INTRODUCTION

The statistical behavior of complex physical systems can be discussed within the framework of statistical mechanics. However the most powerful tools of statistical mechanics are restricted to systems which are in the thermal equilibrium or near the thermal equilibrium where the linear Casimir-Onsager theory is available; see e. g. Kreuzer (1981). If we would like to consider systems far away from the thermal equilibrium only a few analysis concepts are developed. The most prominent systems of this kind are electronic oscillators, lasers and some chemical reactions where so-called limit cycles arise. It is well-known that limit cycles are possible only if nonlinearity as well as energy supply and energy dissipation occur such that these systems are far away from thermal equilibrium. From a statistical point of view the deterministic behavior of this class of systems corresponds to the average behavior and the mathematical concept of dynamical systems can be used that is based on non-linear analysis. Although a generalisation to stochastic dynamical systems is available a suitable relationship to the physical constrains of statistical mechanics does not exist.

On the other hand in classical mechanics of non-linear systems Hamilton’s variational concept can be applied in a successful manner where a close relationship of Hamilton mechanics and the framework of statistical mechanics exists. Unfortunately the Hamilton mechanics is restricted to systems with energy conservation. For this purpose some generalisations of Hamilton mechanics to systems with energy dissipation are known in the literature but these concepts are not close related to Hamilton mechanics if systems with limit cycles are considered. In this paper we will present an alternative generalisation of Hamilton mechanics that includes dissipation and allows limit cycles in a natural way. By means of these so-called canonical dissipative systems (CD systems) a statistical variant can be constructed. We illustrate the concept of stochastic CD systems for the stochastic analysis of a certain class of electronic oscillator circuits.

2. CANONICAL DISSIPATIVE SYSTEMS

The concept of canonical dissipative systems (CD systems) is based on the classical concept of Hamilton systems (Goldstein (1980)) where it is assumed that the energy is preserved. Although the concept was developed for mechanical systems with a finite number of degrees of freedom (DOF) it can be applied to more general physical systems (e. g. electrical circuits) and systems with an infinite number of DOFs, too; e. g. Gossick (1967). Assuming that the number of DOFs is finite we choose a suitable set of variables \((q,p) \in \mathbb{R}^n \times \mathbb{R}^m =: \Gamma\) for a system under consideration, construct the (scalar) Hamilton function \(H(q,p)\) and formulate the dynamics of the system by means of Hamilton’s equations \(\frac{dq}{dt} = \partial_p H, \quad \frac{dp}{dt} = -\partial_q H.\) (1)

The property of energy preservation is indicated by the symplectic structure of Hamilton’s equations that is the r.h.s of these equations can by reconstructed by a symplectic matrix \(J\)

\[
\begin{pmatrix}
\dot{q} \\
\dot{p}
\end{pmatrix} =
\begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix}
\begin{pmatrix}
\partial_q H \\
\partial_p H
\end{pmatrix} =: J
\begin{pmatrix}
\partial_q H \\
\partial_p H
\end{pmatrix}.
\] (2)

If the dimension \(2N\) of the state space (phase space) \(\Gamma\) is rather small we are mainly interested in the trajectories of states \(q = q(t)\) and \(p = p(t)\) where some initial state \((q(t_0), p(t_0))\) at \(t_0\) is given. Especially in multi-particle systems where \(2N\) is very large the initial states are not known very precisely such that it would be much better to use sets of states and study the dynamics of these sets. The idea behind this approach is to consider an ensemble of systems simultaneously. Since it leads to a rather complex
description of systems it is more suitable to use density functions \( f(q, p, t) \) on \( \Gamma \) as states of a system that satisfies
\[
\int f(q, p, t) \, dq \, dp = 1. \tag{3}
\]

If we assume the additional condition \( f(q, p, t) \geq 0 \) the density functions can be interpreted as probability functions such that this concept is called statistical mechanics; see e.g. Tuckerman (2010). Since \( f \) is a conservation quantity the dynamics of states \( f(q, p) \) is induced by Hamilton’s equations, that is \( \partial_t f = \{ H, f \} \),
\[
\partial_t f = \{ H, f \}, \tag{4}
\]
with the so-called Poisson bracket
\[
\{ \cdot, \cdot \} := \sum_{k=1}^{n} \left( \frac{\partial (\cdot)}{\partial q_k} \frac{\partial (\cdot)}{\partial p_k} - \frac{\partial (\cdot)}{\partial p_k} \frac{\partial (\cdot)}{\partial q_k} \right). \tag{5}
\]

If we define the Liouville operator \( \mathcal{L} f := \{ H, f \} \) the dynamical equation can be reformulated \( \partial_t f = \mathcal{L} f \). Especially the equilibrium or steady state behavior of this class of systems can be analysed in a very simple manner. The condition for the steady state behavior is
\[
\partial_t f = 0 \quad \text{such that we have} \quad \mathcal{L} f = \{ H, f \} = 0 \quad \text{that is in essential equivalent with} \quad f = \mathcal{J}(H). \quad \text{The specific form of} \quad f(H) \quad \text{depends on the specific details of the ensemble (microcanonical, canonical or grand canonical ensemble).}
\]

As already mentioned systems with so-called limit cycles cannot described by Hamiltonian equations because dissipativity has to be assumed. In order to generalise the Hamiltonian concept feedback and dissipation are needed. Maschke et al. (2000), Janschek (1967) introduce feedback into a Hamiltonian system by augmenting it by inputs and we obtain the concept of port Hamilton equations such that also dissipative subsystems can be introduced. Another possibility to incorporate dissipativity into the Hamiltonian concept leads the canonical dissipative systems. Following Haken (1973) the dissipativity can be introduced into Hamilton’s equations by means of additive terms which break the symplectic symmetry of Hamilton’s equations. We obtain the diagonal canonical dissipative systems
\[
\begin{pmatrix}
\dot{q} \\
\dot{p}
\end{pmatrix}
= J 
\begin{pmatrix}
\partial_q H \\
\partial_p H
\end{pmatrix}
- 
\begin{pmatrix}
g(H) & 0 \\
0 & g(H)
\end{pmatrix}
\begin{pmatrix}
\partial_q H \\
\partial_p H
\end{pmatrix}, \tag{6}
\]
where \( g(H) \) is denominated as dissipation function. Ebeling and Sokolov (2005) neglected the additive term in the first equation and denoted them as canonical dissipative systems. Obviously the energy is time-dependent
\[
\frac{dH}{dt} = -g(H) \left( \partial_q H \right)^2 + \left( \partial_q H \right)^2. \tag{7}
\]

However the time-dependence disappears if the system is restricted to the zero set of \( g \), that is to all \((q, p) \in \Gamma \) where \( g(H(q, p)) = 0 \).

For a realisation it can be a first step to construct a block structure of a CD system which is shown in fig. 1. This structure can be also a basis to design an associated electronic circuit.

![Fig. 1. Canonical dissipative systems: a block diagram](image)

If in the case of the canonical dissipative systems \( g(H) \) is constant or affine that is \( g(H) = E_0 \) or \( g(H) = -E_0 + H \) we have
\[
\frac{dH}{dt} = -E_0 (\partial_q H)^2 \quad \text{or} \quad \frac{dH}{dt} = -(E_0 - H) (\partial_q H)^2. \tag{8}
\]

In the first case energy is supplied to the system or dissipated from the system in dependence of the sign of \( E_0 \) whereas in the second case energy will be supplied or dissipated if \( H < E_0 \) and \( H > E_0 \), respectively, to or from the system; see fig. 2. Confining to canonical dissipative systems with a limit cycle the dissipation function \( g \) have to be chosen in such a manner that the zero set \( Z := \{ (q, p) \in \Gamma | g(H(q, p)) = 0 \} \) is a close trajectory in \( \Gamma \). Obviously if we consider trajectories of eq. (6) where the initial state is in \( Z \) the trajectories will be remain in \( Z \) for all \( t > t_0 \).

As already mentioned it is possible to formulate Hamilton’s equations also for electronic circuits. Using the generalised variables charge \( q \) (at the capacitor) and flux \( \phi \) (through the inductor) Hamilton’s equation can be constructed a canonical dissipative circuit where we assume a nonlinearity that leads to a affine dissipation function \( g \). It is known that the so-called Rayleigh-van der Pol equation is probably the simplest case of a CD system. The corresponding circuit equations in a CD form can be given as follows
\[
\frac{dq}{dt} = \frac{\phi}{L}. \tag{9}
\]
The corresponding phase portrait of the Rayleigh-van der Pol systems can be reduced as a second order differential equation by using $q = C u$

$$\frac{d^2u}{dt^2} + \frac{1}{LC} u = - \left( \frac{1}{2} C^2 \left( \frac{du}{dt} \right)^2 + \frac{C}{2L} u^2 - E_0 \right) \frac{du}{dt}. \quad (11)$$

If the bracket term on the r.h.s. is positive constant – that is we have a constant dissipation function $g$ – we obtain a linear RLC circuit. In order to interpret this equation as an electronic circuit it can be shown that a non-linear resistor and a non-linear controlled source is needed. An example of such a circuit is shown in fig. 4. Another interesting example is the so-called Cassini oscillator; see e. g. Rodriguez et al. (2011). Its Hamiltonian is given by

$$H(q,p) = ((p-1)^2 + q^2)((p+1)^2 + q^2) \quad (12)$$

such that the CD equations of the Cassini oscillator can be formulated for an affine dissipation function $g$

$$\frac{dq}{dt} = 4p(1 + q^2 + p^2), \quad (13)$$

$$\frac{dp}{dt} = 2(p^2 + (-1 + q)^2)(1 + q) - 2(-1 + q)(p^2 + (1 + q)^2) - 4(p(1 + q^2 + p^2) \cdot \cdot \cdot \cdot \cdot (1 - E_0 + 2p^2 + p^4 + 2(-1 + p^2)q^2 + q^4). \quad (14)$$

A direct circuit interpretation is not possible but a circuit realization can be derived by using the block diagram at the system level. However it is more interesting to have a look at the zero set of the dissipation function $g(H)$ of the Cassini oscillator. It depends on the energy parameter $E_0$. In fig. 6 the zero sets are shown for different values of $E_0$. By means of the dissipation function as surface in fig. 5 the zeros sets can be interpreted as intersections.
CD systems dissipation and fluctuation coexist. This leads to the well-known dissipation-fluctuation theorem. As a result we obtain a stochastic reaction to the CD system that can be studied in a simple manner if the CD system is augmented by an additive white noise term. We obtain the descriptive equations of stochastic diagonal canonical dissipative systems

\[
\begin{pmatrix}
\dot{q} \\
\dot{p}
\end{pmatrix} = J \begin{pmatrix}
\partial_q H \\
\partial_p H
\end{pmatrix} - \begin{pmatrix} 0 \\
g(H) \partial_p H
\end{pmatrix} + \begin{pmatrix} 0 \\
\sqrt{2D(H)} \xi(t)
\end{pmatrix},
\]

where \(\xi(t)\) is a white noise stochastic process. The fluctuation function \(D(H)\) depends just like the dissipation function \(g(H)\) only on the Hamilton function \(H\). Recently Thiessen and Mathis (2010) has shown that certain noise aspects of electronic oscillators can be studied by using stochastic CD systems.

From the mathematical point of view it is a second order system of stochastic differential equations (SDE). A direct solution of this system of SDEs is not possible but under if the solution is a Markov process a corresponding Fokker-Planck equation for the probability density \(W(q,p,t)\) can be constructed. Using the Stratonovich approach we obtain after some calculations where \(W\) can be interpreted as a function on \(\Gamma\)

\[
\frac{\partial W}{\partial t} = -\{H,W\} + \frac{\partial}{\partial p} (g(H)\partial_p H W + D(H)\partial_p W).
\]

The complete Fokker-Planck equation as well as the stationary Fokker-Planck equation \((\partial W/\partial t = 0)\) is not solvable in an analytical manner. But the ensemble concept for CD systems motivates to consider a subclass of solutions of the stationary Fokker-Planck equation (22) where \(W_{st} = W_{st}(H)\) is satisfied. We obtain the following condition results from eq. (22)

\[
-g(H)\partial_p W_{st}(H) = D(H) \frac{\partial W_{st}(H)}{\partial H} \partial_p H.
\]

This ordinary differential equation of first order can be solved

\[
W_{st}(H) = \frac{1}{Q} \exp \left( - \int_0^H \frac{g(\tilde{H})}{D(\tilde{H})} d\tilde{H} \right),
\]

with

\[
Q := \int_{-\infty}^{\infty} W_{st}(H) dH.
\]

As an example we consider a linear RLC circuit with the Hamilton function \(H(q,p) = \phi^2/2 + q^2/2\) (with a suitable scaling) and with a constant dissipation function \(g(H) = 1/R\) and a constant fluctuation function \(D(H) = D\). In this case we have

\[
W_{st}(H) = \frac{1}{Q} \exp \left( - \int_0^H \frac{1}{DR} d\tilde{H} \right) = \frac{1}{DR} \exp \left( -\frac{H}{DR} \right).
\]

where the probability density is the well-known exponential function and in accordance with the famous fluctuation dissipation theorem.
Fig. 8. A simple canonical dissipative circuit with \( g(H) = 1/R \)

If we consider the Rayleigh-van der Pol equation with its affine dissipation function \( g(H) = \beta(H - E_0) \) and a constant dissipation function \( D(H) = D \). Using formula (24) for \( W_{st} \) we obtain

\[
W_{st}(H) = \frac{1}{Q} \exp\left(\frac{\beta H(2E_0 - H)}{2D}\right)
\]

(27)

where the second representation was obtained by varying the constant \( Q \). This probability density \( W_{st}(q, \phi) \) over the \( q, \phi \)-plane is shown in fig. 9. The probability density function \( W_{st} \) is a non-Gaussian density where its maximal values correspond with the (known) limit cycle of the Rayleigh-van der Pol equation. It should be mentioned that the deviation of the density from the Gaussian density is no problem because their is a coupling of the oscillatory system with an energy reservoir. Even in the equilibrium thermodynamic a Gaussian density is not necessary. Furthermore the expectation value of the energy is \( \langle H \rangle = \int H W_{st}(H) dH = E_0 \) such that it is related only with the dissipation function \( g \). However the energy fluctuation

\[
\delta E^2 = \langle (H^2) \rangle - \langle H \rangle^2 = D
\]

(29)

is determined by the fluctuation function \( D(H) = D \). Since the stationary solution of the Fokker-Planck equation satisfies Boltzmann’s H-theorem it seems that all these properties suggest a physical structure behind them. On the other hand there is no relationship between \( g(H) \) and \( D(H) \) such that any kind of fluctuation dissipation theorem is missing. Moreover it should be mentioned that Weiss and Mathis (1998) (see also Mathis and Weiss (2003)) had shown that the Fokker-Planck concept has to be extended by using the Kramers-Moyal series (see e.g. van Kampen (2007)) if non-linear reciprocal circuits are studied near the thermal equilibrium. The validity of their approach is shown in Weiss and Mathis (1999).

As final example we consider the Cassini oscillator with Hamiltonian (12) we obtain the following probability density for an affine dissipation function \( g(H) = H - E_0 \)

\[
W_{st}(H) \sim \exp\left(-\frac{(p - 1)^2 + q^2}{2D}\right) \frac{1}{\beta} \exp\left(-\frac{\beta(H - E_0)^2}{2D}\right)
\]

(30)

The probability density \( W_{st} \) is shown in figs. 10, 11 for most interesting cases where two separate limit cycles are appear with energies \( E_0 \) and the fluctuation intensities \( D \). In order to verify our analytical results we solve the stochastic differential equations (21) by means of the Euler-Maruyama integration method; see e.g. Kloeden et al. (2003) where some results are shown in fig. 12 and fig. 13. In fig. 12 the two limit cycles are distant and the stochastic trajectory converges to one of the limit cycles where a transient occurs. In contrast to that behavior the stochastic trajectory in fig. 13 moves from one limit cycle to the other since there is a non-vanishing probability for changing the limit cycle; see fig. 11. This effect can be very interesting for future electronic circuits; e.g. it can be used to develop a certain class of coding systems with oscillatory signals.

4. CONCLUSION

In this paper it is shown that Hamilton’s concept for the description of systems with energy conservation can be
extend in such a manner that dissipation can be included. This class of systems is called canonical dissipative systems (CD systems). In this paper we study a certain subclass of CD systems where limit cycles appear. In order to include indeterminacy with respect to the initial states ensemble concepts from statistical mechanics are discussed. Since dissipation is related to fluctuation on the microscopic level a stochastic white noise term has to be added such that stochastic differential equations arise. We discuss an analytical method using the Fokker-Planck equation as well as a numerical validation method for the calculation of the probability density. The described methods are illustrated by means of several examples from electronic circuits. Using our approach a new and alternative approach for the analysis of certain noise aspects of electronic oscillators is presented. Since phase noise of oscillators is a main subject in electronic circuit design it is an essential problem to incorporate phase noise into our considerations. Furthermore since most of electronic oscillators do not belong to the class of CD systems an approximation method based on CD systems should be developed. First results are already obtained such that there is some hope to be successful in the future.

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REFERENCES