Optimal Control of Motion of a Robot
Driven by a Movable Internal Body in a
Resistive Environment

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Abstract: A two-body system moving along a horizontal line in a nonlinear viscous medium
is considered. One of the bodies (main body) interacts with the environment and with the
other body (internal body), which interacts with the main body but does not interact with the
environment. A periodic optimal motion of the internal body relative to the main body, which
sustains the motion of the main body with periodically changing velocity and maximizes its
average speed, is defined by solving an optimal control problem.

Keywords: Mobile robots, resistive media, nonlinear systems, periodic motion, optimal control.

1. INTRODUCTION

An optimization problem for the motion of a body that
is controlled by its interaction with a movable internal
body and moves in a resistive environment was first stated
by Chernousko (2005, 2006). The case where the main
body moves along a straight line on a horizontal plane and
is acted upon by Coulomb’s dry friction was considered.
Periodic control modes were constructed for the relative
motion of the internal body, such that the main body
moves with periodically changing velocity. The internal
body is allowed to move within fixed limits. Velocity-
controlled and acceleration-controlled modes were consid-
ered for the motion of the internal body. In the former
case, the internal body moves between the fixed extreme
positions with constant velocity relative to the main body.
The parameters to be varied here are the magnitudes of
the velocities of motion of the internal body. In the latter
mode, each period has three intervals on which the relative
acceleration of the internal body is constant. A constraint
is imposed on the absolute value of this variable. A periodic
control with zero average and the corresponding motion
of the main body with periodically changing velocity and
maximum displacement for the period were constructed.
The control constructed allows one to restore the periodic
law of motion of the internal body that generates the op-
solved a similar problem for a system with two internal
bodies, one of which moves periodically along a horizontal
line, parallel to the line of motion of the main body, while
the other moves along a vertical line. The internal body
moving along the vertical enables one to control the normal
pressure of the main body on the supporting plane and, as
a consequence, the force of friction that acts on the main
body when it is moving.

Egorov and Zakharova (2010) constructed and investi-
gated energy-optimal control modes for the system with
one internal body moving in the environments with power-

law resistance to the motion of the main body. The energy
consumption was characterized by the work produced by
the resistance force for the period of motion of the system.
When constructing the optimal control, the period of the
relative motion of the internal body and the average ve-
locity of the system were prescribed; no other constraints
were imposed on the motion of the system.

2. STATEMENT OF THE PROBLEM

A two-body mechanical system moving in a resistive
environment is considered. The system consists of the
main body that interacts with the environment and an
auxiliary internal body that interacts with the main body
but does not interact with the environment. Both bodies
move translationally along the same straight line. The motion of the system is excited and controlled by the motion of the internal body relative to the main body. Let \( x \) denote the displacement of the main body relative to the fixed environment, \( \xi \) the displacement of the internal body relative to the main body, \( M \) the mass of the main body, \( m \) the mass of the internal body, and \( R(\dot{x}) \) the resistance force exerted by the environment on the moving main body. The motion of the system is governed by the differential equation

\[
(M + m)\ddot{x} = -m\ddot{\xi} + R(\dot{x}).
\]

For this system, we will seek a \( T \)-periodic motion of the internal body \( \xi(t) \) constrained by \( |\dot{\xi}| \leq A \), where \( A \) is a given positive quantity, and the corresponding motion of the main body \( x(t) \) such that its velocity \( \dot{x}(t) \) is \( T \)-periodic and the motion for the period is a maximum. The period \( T \) is fixed.

Introduce the dimensionless variables

\[
v = \frac{M + m}{mAT} \dot{x}, \quad w = \frac{\dot{\xi}}{AT}, \quad u = \frac{\ddot{x}}{\frac{m}{M + m}v}, \quad \bar{t} = \frac{t}{T},
\]

\[
r(v) = -\frac{1}{mA} R\left(\frac{mAT}{M + m}v\right)
\]

to state the optimal control problem for the system under consideration as follows.

**Problem.** For the system

\[
\dot{v} = -u - r(v), \quad \dot{w} = u
\]

find a control \( u(t) \) that satisfies the constraint

\[
|u| \leq 1
\]

and maximizes the performance index

\[
J = \int_0^1 v(r) d\tau,
\]

subject to the conditions

\[
v(0) = v(1), \quad w(0) = w(1).
\]

The boundary conditions (6) are implied by the periodicity of the functions \( v(t) \) and \( \xi(t) \).

We assume that the function \( r(v) \) is continuous, vanishes for \( v = 0 \), monotonically increases, does not have linear segments, and is twice differentiable everywhere, except, possibly, for the point \( v = 0 \), i.e.,

\[
r(0) = 0, \quad r'(v) \geq 0; \quad r''(v) \neq 0 \text{ for any interval}
\]

From the relations (3) and (6) it follows that \( \int_0^1 r(v(t)) dt = 0 \), which implies that the function \( v(t) \) changes in sign or is identically zero.

The solution of the optimal control problem possesses a number of important properties that are stated by the following propositions (proofs are omitted).

**Proposition 1.** For any admissible motion, \( |r(v(t))| < 1 \).

**Proposition 2.** For the resistance law \( r(v) \) satisfying the conditions of (7), there exists an admissible control for which the displacement of the main body for the period is not equal to zero.

**Corollary.** If \( r(v) \) is an odd function that satisfies the conditions of (7), then there exists an admissible control for which the displacement of the main body for the period is positive.

**Proposition 3.** Any motion with a \( 1/n \)-periodic velocity, which is not identically zero, for an integer \( n > 1 \) is not optimal.

### 3. Maximum Principle Boundary-Value Problem

Apply the maximum principle to Problem and represent the maximum principle boundary-value problem as follows:

\[
\dot{v} = -u(v, s) - r(v), \quad \dot{s} = (s - c)r'(v) + 1, \quad v(0) = v(1), \quad s(0) = s(1),
\]

where

\[
u = \begin{cases} 
sign s, & s \neq 0, 
-\dot{r}(v), & s = 0. 
\end{cases}
\]

The variable \( s \) and the constant \( c = \text{const} > 0 \) are expressed as \( s = c - p_v c \), \( p_v \) are the variables \( p_v \) and \( p_{\dot{v}} \) are adjoined to \( v \) and \( w \), respectively. The definition \( u = -\dot{r}(v) \) for \( s = 0 \) does not violate the constraint \( |u| \leq 1 \), by virtue of Proposition 1.

The Hamiltonian has the form

\[
H = s(v + r(v)) - cr(v) + v.
\]

The following relations are valid on a singular segment:

\[
s \equiv 0, \quad cr'(v) = 1, \quad v = \text{const}, \quad r''(v) \geq 0.
\]

For \( r''(0) > 0 \), the maximum principle boundary value problem has the trivial singular solution

\[
v(t) \equiv 0, \quad u(t) \equiv 0, \quad s(t) \equiv 0, \quad c = 1/r'(0).
\]

The range of the function \( v(t) \) that corresponds to the optimal motion and is not identically zero is an interval \([v_-, v_+]\), where \( v_- < 0 < v_+ \). At the time instants when \( v = v_- \) or \( v = v_+ \) the variable \( s \) must vanish. Since the system of equations (8) and (11) is time-invariant, the Hamiltonian (12) is constant along an optimal trajectory. Moreover, the Hamiltonian is a first integral of this system of equations. We will take as the initial instant the instant at which the velocity of the main body \( v(t) \) is equal to its maximum value, \( v(0) = v_+ \), the equality relation \( s(0) = 0 \) also occurs at this instant. The parameters \( c \) and \( v_+ \) are the desired parameters of the boundary value problem.

Since the Hamiltonian (12) is constant and \( s(0) = 0 \), it follows that for \( v_+ \) and \( c \) solving the maximum principle boundary value problem, the relation

\[
s(u + r(v)) = cr(v) - v - (cr(v_+) - v_+)
\]

holds. The left-hand side of this relation is nonnegative. Hence, its right-hand side is also nonnegative and vanishes.
if and only if $s = 0$. It was proved that $c > 0$. Taking into account this fact we get the relations

$$r(v) \geq l(v), \quad r(v) = l(v) \Leftrightarrow s = 0,$$

where $l(v) = \frac{v - v_+}{c} + r(v_+)$$

which hold in the interval $[v_-, v_+]$. Geometrically, this means that in the interval $[v_-, v_+] \ni 0$ the graph of the function $r(v)$ lies above or on the straight line $l(v)$. For $v = v_-$ and $v = v_+$, the graph and the straight line intersect, in particular, are tangent to each other. If $v_0 \in [v_-, v_+]$ is a point of tangency of $r(v)$ and $l(v)$, then $cr'(v_0) = 1$. This relation coincides with the necessary condition for the singular mode (13), and hence, the motion in a singular mode is possible for the values of the velocity $v$ that correspond to the tangency of $r(v)$ and $l(v)$, and only for these values.

Notice that the inequality

$$r''(v_0) \geq 0$$

holds for the point $v_0$. Otherwise, the function $r(v)$ would have been convex upward at the point $v_0$, and the relation $r(v) < l(v)$ would have taken place in a neighborhood of this point. The inequality (17) is a necessary condition for the singular mode with the velocity $v_+$.

If the curve $r(v)$ and the straight line $l(v)$ intersect transversally (being not tangent to each other) at the point $v_+ = v_-$, then on the arrival at this point, the control $u$ instantaneously switches from $-1$ to $1$ or from $1$ to $-1$, respectively. If $v_+ = v_-$ is a point of tangency of $r(v)$ and $l(v)$, then on the arrival at this point, the control can instantaneously switch from $-1$ to $1$ or from $1$ to $-1$ or it can switch from $-1$ to the singular value $u = -r(v_+)$ or from $1$ to the singular value $u = -r(v_-)$, respectively. After a while, the system leaves the singular mode with the control $u = 1$ at the point $v_+$ or $u = -1$ at the point $v_-$. If $v_0$ is a point of tangency of $r(v)$ and $l(v)$ in the interval $(v_-, v_+)$, then the following scenarios are possible for the behavior of the system in the neighborhood of such a point.

The system arrives at the point $v_0$, either from the domain $v < v_0$ with the control $u = -1$ or from the domain $v > v_0$ with the control $u = 1$. At the point $v = v_0$, the control either takes on for an instant the value $u = -r(v_0)$, in accordance with relation (11), or the system enters the singular mode with the control $u = -r(v_0)$, which lasts for some time $\tau$. Then the system continues moving either into the domain $v < v_0$ with the control $u = -1$ or into the domain $v > v_0$ with the control $u = 1$.

Between the points of intersection of the curve $r(v)$ and the straight line $l(v)$, the system can move only with $u = 1$ or $u = -1$ without control switching.

For an arbitrary function $r(v)$, constrained only by conditions (7), the solution of the optimal control problem is complicated because of plurality of possible scenarios of the system’s behaviour. For this reason, we confine ourselves to the functions $r(v)$ such that any straight line that intersects the graph of the function at the points $v_-$ and $v_+$ such that $v_- < 0 < v_+$ and lies below or on the curve $r(v)$ in the interval $[v_-, v_+]$ does not have common points with this curve in the interior of the interval. Define class $\mathcal{R}$ of functions $r(v)$ that possess such a property.

4. FUNCTIONS OF CLASS $\mathcal{R}$

Definition. A function $r(v)$ belongs to $\mathcal{R}$ if this function satisfies conditions (7) and for any tangent to the curve $r(v)$ at a point at which this curve is convex downward, there do not exist two points of intersection of the curve and the tangent, one of which lies on the left of the point of tangency and the axis of ordinates, while the other lies on the right of the point of tangency and the axis of ordinates.

Class $\mathcal{R}$, for example, contains functions $r(v)$ that satisfy conditions (7), are convex upward in some interval $[a_1, a_2]$, and are convex downward outside this interval. This is the case, in particular, for the functions $r(v)$ that have one inflection point and for the functions $r(v)$ that are convex upward (downward) everywhere.

In what follows, we consider only functions $r(v)$ of class $\mathcal{R}$. In this case, for any $v_+$ and $c$ such that $v_- < 0 < v_+$, the relations $r(v) > l(v), s \neq 0$ and $u = \pm 1$ hold for $v \in (v_-, v_+)$. At the points $v_\pm$, the control switches, $s = 0$, and the system may proceed to a singular mode, if $r''(v_\pm) = 1/c, r''(v_\pm) \geq 0$.

Proposition 4. If $r(v) \in \mathcal{R}$, in the optimal motion the relations $u = 1, \dot{v} = -1 - r(v)$ and the relations $u = -1, \dot{v} = 1 - r(v)$ are fulfilled each in only one interval from $[0, 1]$. (The proof is based on Proposition 3.)

Proposition 5. If $r(v)$ is an odd function of class $\mathcal{R}$, then the optimal control has a singular segment.

5. SOLUTION OF THE BOUNDARY-VALUE PROBLEM FOR SOME FAMILIES OF RESISTANCE LAWS FROM $\mathcal{R}$

5.1 Solution for the functions $r(v)$ satisfying the inequality $r(v) > mu \neq 0, \mu > 0$

In particular, this inequality is satisfied for $\mu = r''(v) = 0$. For any such resistance laws, the inequality $r(v) > |r''(v)|$ is valid for any positive $\mu$. The only solution of the maximum principle boundary value problem is the trivial solution (14), so it is impossible to construct an admissible motion, in which the main body would shift right for the period.

5.2 Solution for the case of $r''(v) < 0$

In this case, a singular mode is impossible and $u = \text{sign}(1/2 - t)$. The parameters $v_-$ and $v_+$ can be determined by solving the system of equations $\int_{v_-}^{v_+} dv/(1 \pm r(v)) = 1/2$. This solution exists and is unique.

5.3 Solution for the isotropic resistance satisfying the relation sign $r''(v) \text{sign} v = \text{const}$

In particular, this property is possessed by a medium with power-law resistance $r(v) = k|v|^{a-1} \text{sign} v, a > 0$. For $r''(v) \text{sign} v > 0$, the singular velocity is $v_+$, the relation $c = 1/r''(v_+)$ is valid, and the inequality $|v_-| > v_+$ holds.
Similarly, for \( r''(v) \text{sign} v < 0 \), the singular velocity is \( v_- \), the relation \( c = 1/r'(v_-) \) and the inequality \( |v_-| < v_+ \) being valid.

Let \( r''(v) \text{sign} v > 0 \). For the power-law resistance, this inequality holds for \( \alpha > 1 \). Then the optimal control \( u(t) \) is given by

\[
u(t) = \begin{cases} 
1, & t \in [0,t_1), \\
-1, & t \in (t_1,t_2), \\
-r(v_+), & t \in [t_2,1]
\end{cases}
\] (18)

The velocity \( v(t) \) obeys the relations \( v(0) = v(t_2) = v_+, \; v(t_1) = v_- \). The unknowns \( t_1, t_2, v_-, v_+ \) are defined by the system

\[
t_1 = \int_{v_-}^{v_+} \frac{dv}{1 + r(v)}, \quad t_2 = 2 \int_{v_-}^{v_+} \frac{dv}{1 - r^2(v)}
\] (19)

\[
2 \int_{v_-}^{v_+} \frac{r(v_+) - r(v)}{1 - r^2(v)} dv = r(v_+),
\] (20)

\[
r(v_-) - r(v_+) - r'(v_+)(v_- - v_+) = 0
\] (21)

6. CONCLUSION

A mechanical system that consists of the main body, interacting with a resistive environment, and an internal body that interacts with the main body but does not interact with the environment can move with periodically changing velocity and nonzero displacement for the period in any medium with nonlinear law of resistance to the motion of the main body.

If the medium is isotropic, then the motion can be implemented in any direction.

If the force of resistance of the environment to the motion of the main body is characterized by a convex downward function of the velocity of this body, then the motion with nonzero average velocity in the positive direction is impossible. If this force is characterized by a convex upward function of the velocity of the main body, then the optimal control alternates the intervals in which the control variable takes on its maximum and minimum allowed values, the duration of the intervals of each type being equal to half of the period.

If the resistance law is characterized by an odd function of the velocity of the main body, the sense of convexity of which is constant while the velocity is constant in sign, then the optimal motion involves nonsingular and singular segments. A singular segment follows each pair of nonsingular segments, in which the control variable takes on its maximum or minimum allowed values and the main body moves with variable velocity. In the singular segments, the control variable takes on a constant value, intermediate between the extreme allowed values, and the main body moves with a constant velocity. The segments of each type are repeated in a period.

REFERENCES


