Bond Graphs and Lagrange Equations as Aids in Analytical Studies of Electro-Mechanical Systems

Dean Karnopp

dckarnopp@ucdavis.edu
Department of Mechanical and Aerospace Engineering
The University of California, Davis
One Shields Ave, Davis, CA 95616

Abstract: A well-known advantage of bond graph models of electro-mechanical systems is that they can be manipulated manually or with the aid of computer programs to yield first order state equations useful for numerical studies. A causal analysis can show in advance whether derivative causality will cause practical difficulties in the equation formulation which might be best circumvented by modification of the model.

Numerical studies are sometimes of limited use in preliminary investigation of the feasibility of a design because the large number of parameters involved precludes a general understanding of the limits of the concept being studied. Analytical studies of simplified models may find the existence of parameter combinations which are crucial to the successful operation of the system or fundamental limitations to performance not obvious in purely numerical studies.

Although the first order nature of the normal bond graph state equations are advantageous for numerical simulation, they are not necessarily the most convenient form for analytic studies using pencil and paper. In many cases, the use of Lagrange equations, based on energy state functions and degrees-of-freedom rather than state variables are advantageous for analytic work. Bond graph models and a special form of causality can be used to advantage, particularly when derivative causality is present, to yield sets of second order dynamic equations based on Lagrange equations, which are often easier to work with than their first order equivalents. This will be illustrated by an example by the analysis of the use of a rotary electric motor as an actuator in semi-active or active suspension element in a vehicle suspension.

Keywords: Bond graphs, Lagrange equations, mechatronic systems

1. INTRODUCTION

Bond graphs are based on power and energy interactions and so too are Lagrange equations. It is obvious then, that there should connections between the two techniques of system modeling and analysis. The original and most common use of Lagrange equations is for mechanical systems, Karnopp and Margolis (2007), but it has long been appreciated that these equations also apply to electrical and electromechanical systems, Wells (1938), Crandall et. al. (1968). Since bond graphs are based on analogous variables in a wide variety of physical systems, the state functions associated with I- and C-elements can be used to derive the standard Lagrange equations for any physical system represented by a bond graph, Karnopp et al. (2005), Karnopp (1977).
The standard first order bond graph state equations involve the rates of change of momenta, \( \dot{p} \), for I-elements and, and displacements, \( \dot{q} \), for C-elements. The second order Lagrange equations involve the second time derivatives of generalized displacement coordinates, \( \ddot{q} \). For most system models, there are twice as many first order state equations as there are second order Lagrange equations.

It is easy to write down the general form of the Lagrange equations that we will be using but each term requires some explanation and a procedure applicable to bond graphs needs to be formulated. The equations, derived from a generalized Hamilton’s principle, \[3\], are

\[
\frac{d}{dt} \frac{\partial T^*}{\partial \dot{q}_i} - \frac{\partial T^*}{\partial q_i} + \frac{\partial V}{\partial q_i} = E_i, \quad i = 1, 2, ..., n. \tag{1}
\]

These equations apply to a holonomic system described by a set of \( n \) generalized displacement variables \( q_i \) and their flows \( f_i = \dot{q}_i \). The system is said to have \( n \) (displacement) degrees of freedom.

The symbol \( T^* \) represents the complementary energy for I-elements. In mechanical systems this quantity is called kinetic coenergy and in electrical system it is magnetic coenergy. The distinction between energy and coenergy is crucial for nonlinear systems and in any case \( T^* \) is properly a function of the flow variables \( \dot{q}_i \), Crandall et al. (1968), Karnopp et al. (2005), Karnopp (1977). The symbol \( V \) is the energy for C-elements. It is a function of the generalized displacements \( q_i \) and represents potential energy in mechanical systems and electrical energy in electrical systems. The symbol \( E_i \) represents a generalized effort due to all bond graph elements except the I- and C-elements already represented in \( T^* \) and \( V \). It could represent forces or moments or voltages applied by effort sources or R-elements.

As outlined in Karnopp (1977), the required number of displacement degrees of freedom \( n \) can be determined by applying artificial or virtual flow sources to the model and sequentially extending the causality in the bond graph until all bonds are causally augmented. Each virtual source will define a generalized flow \( \dot{q}_i \) and a generalized displacement \( q_i \). The geometric compatibility relations will be satisfied only when the constants of integration from generalized flow to generalized displacement are properly chosen. This process can be applied to any bond graph but will be illustrated using an example involving a controlled electromechanical shock absorber device to be used in active or semi-active vehicle suspensions.

2. SEMI-ACTIVE DAMPERS

Figure 1 shows one version of an electromechanical shock absorber consisting of a rotary dc motor and a rack and pinion arrangement. If the motor is connected to an electronically variable resistance element, the device becomes a linear semi-active damper with an easily controlled damping coefficient. (If it is connected to an amplifier, it becomes an active suspension element.) Here we will study the device’s use as a semi-active element. The attempt is to discover the limitations of the motor when used in this fashion.

Figure 2 shows an elementary “quarter car” model to illustrate the advantages of a damper, \( B_s \), reacting to the mass absolute velocity, \( \dot{z}_s \), rather than a damper, \( B \), reacting to the relative velocity, \( \dot{z}_r - \dot{z}_s \). Of course, it is normally not possible to
realize the so-called “skyhook damper” $B_s$, exactly
but it has been shown theoretically and in practice
that a semi-active damper, $B$, in the location shown
can be programmed to simulate the action of the
skyhook damper enough of the time to achieve
significant advantages compared to a damper with
fixed characteristics, Karnopp (1995), Karnopp
(1983). Although linear motors have been proposed
to realize an active damper, Karnopp (1989), the use
of a rotary motor requires a more complex dynamic
model including the moment of inertia of the motor.

The bond graph in Figure 2 shows that if a single
virtual flow source, (shown with a dotted bond), is
inserted according to the method in Karnopp (1977),
and causality is extended, all bonds become causally
augmented. Note that the C-element is in integral
causality while the I-element is in derivative
causality. This means that the system has a single
placement degree of freedom with generalized
displacement $z_s$.

The coenergy for the I-element is simply
$T^* = (1/2)M z^2$, and the energy for the C-
element is $(1/2)K(z^2_r - z^2_s)$, where the sketch
implies that $z_s$ and $z_r$ are deviations from
equilibrium positions so that the spring is unstretched
when both displacements are zero.

The generalized effort $E_s$ appears on the virtual
source bond taking account the efforts imposed by
the R-elements but not the I- and C-elements. The
result is $E_s = -B_s z_s + B(z^2_r - z^2_s)$. Using
Eq.(1) the single Lagrange equation is

$$M \ddot{z}_s + (B + B_s) \dot{z}_s + K z_s = B \dot{z}_r + K z_r. \quad (2)$$

The transfer function corresponding Eq. (2) is

$$\frac{z_r}{z_s} = \frac{B_s + K}{M s^2 + (B + B_s)s + K} \quad (3)$$

3. ADVANTAGES OF THE SHYHOOK DAMPER

For an ideal skyhook damper suspension, $B = 0$,
and the transfer function has only a simple
numerator, $K$. The advantage of this can be
appreciated by considering the frequency response
function by setting $S = j \omega$ with $\omega$ representing
the circular forcing frequency. Near the natural
frequency, $\omega_n = \sqrt{K / M}$ the system exhibits a
resonance which can be reduced by making $B_s$
sufficiently large. For example the damping ratio

$$\zeta = \frac{1}{2} B_s \sqrt{K M}$$
can be given a value near unity. This has no effect on the high frequency
response which falls off as $(K / M) / \omega^2$
independent of the value of $B$. Thus the suspension
acts nearly as an ideal low pass filter with a break
frequency at the natural frequency which is generally
a desirable aspect of such a suspension.

In contrast, a conventional system with
$B_s = 0$, $B \neq 0$ has more compromises.

Controlling the resonant peak is possible by adjusting
$B$ but this affects the high frequency response
which falls off as $(B / M) / \omega$. Thus
reduction of the resonant peak and the fall off of the
high frequency response are antagonistic goals when
considering the proper value of $B$. Furthermore,
there is a fixed point in the frequency response. At
$\omega = \sqrt{2K / M}$, $|z_s / z_r| = 1$ independent of the value of $B$.

4. THE USE OF A ROTARY
D.C. MOTOR AS A SEMI-ACTIVE DAMPER

Figure 3 shows a model of Figure 2 using a semi-
active damper based on the type of device shown in
Figure 1. The motor has a moment of inertia, $J$, a
transduction coefficient, $T$, and a variable
resistance, $R$, which is a combination of $R_m$, the
resistance of the motor coils and the controllable
external resistance which can vary from zero to
infinity. Thus $R_m \leq R \leq \infty$. Another parameter
of interest is the radius of the pinion, $r$, which is the
transformer modulus relating the motor angular
velocity to the rack relative velocity. The model
neglects other motor parameters such as the coil
inductance.

![Figure 3 System Using a Rotary DC Motor as a Semi-Active Damper]
the causal marks on the bond graph,
\[ T^* = (1/2)M\ddot{z}_s^2 + (1/2)[J(\dot{z}_r - \dot{z}_s)^2 / r^2] \] and
\[ V = (1/2)K(z_r - z_s)^2. \]

The generalized effort is found by computing the effort on the virtual bond due to the resistive elements, \( E_s = -B_s\dot{z}_s + (T^2 / Rr^2)(\dot{z}_r - \dot{z}_s). \)

The Lagrange equation is then
\[ (M + J / r^2)\ddot{z}_s + (B_s + T^2 / Rr^2)\dot{z}_s + Kz_s, \quad (4) \]
\[ = J / r^2\dot{z}_r + (T^2 / Rr^2)\dot{z}_s + Kz_s, \]

and the transfer function between the suspended mass motion and the input motion is
\[ \frac{z_s}{z_r} = \frac{(J / r^2)s^2 + (T^2 / Rr^2)s + K}{(M + J / r^2)s^2 + (B_s + T^2 / Rr^2)s + K} \quad (5) \]

From this result, one can deduce several facts. First, the rotary inertia does affect the system response. The importance can be judged to some extent by comparing an equivalent mass \( m_{eq} = J / r^2 \) to \( M. \)

If \( m_{eq} = J / r^2 \ll M \), the effect of the rotary inertia will be small. Also, comparing Eqs. (3) and (5) one can see that an equivalent damping parameter can be defined.

\[ B_{eq} = T^2 / Rr^2. \quad (6) \]

The maximum value of \( B_{eq} \) is when \( R \) has its minimum value of \( R_m \) when the motor terminals are shorted, \( B_{eq,\text{max}} = T^2 / R_m r^2. \) On the face of it, this appears to mean that any motor can be used to produce a maximum desired damping coefficient by picking a small enough value of the radius \( r. \)

Practically speaking, however, very small values of \( r \) will mean not only very high stresses and an amplified effect of friction forces, so far neglected in the model, but also will increase the effect of the rotary inertia term. Using Eq.(6), we find that
\[ m_{eq} = J / r^2 = JB_{eq,\text{max}} R_m / T^2, \quad r = \sqrt{T^2 / R_m B_{eq,\text{max}}}. \quad (7) \]

The equations in Eq. (7) allow one to determine whether the motor inertia will be important given moment of inertia, the desired maximum damping coefficient, the coil resistance and the transduction coefficient. One can also determine whether an impractically small value of the pinion radius would be required.

This elementary example shows how simple models of electromechanical systems can show how various parameter combinations can be combined to indicate whether certain systems will be practical. We now show how a Lagrange approach can be particularly useful for higher order models.

5. A TWO DEGREE-OF-FREEDOM MODEL

Figure 4 shows a two degree-of-freedom model using a controllable electromagnetic semi-active damper. The bond graph shows two virtual flow sources imposing \( \dot{z}_s \) and \( \dot{z}_u \) as generalized flows and implying \( z_s \) and \( z_u \) as generalized displacements. Now,
\[ T^* = 1/2(M\ddot{z}_s^2) + 1/2(m\ddot{z}_u^2) + 1/2[J(\dot{z}_u - \dot{z}_s)^2 / r^2], \]
\[ V = 1/2 K(z_u - z_s)^2 + 1/2 K_i(z_r - z_u)^2. \]

There are now two generalized efforts appearing of the two virtual bonds:
\[ E_s = -B_s\dot{z}_s + (T^2 / Rr^2)(\dot{z}_u - \dot{z}_s) \] and
\[ E_u = -(T^2 / Rr^2)(\dot{z}_u - \dot{z}_s). \]

The two Lagrange equations (in the s-domain) are
\[ \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} z_s \\ z_u \end{bmatrix} = \begin{bmatrix} 0 \\ K_i \end{bmatrix} \begin{bmatrix} \dot{z}_s \\ \dot{z}_u \end{bmatrix}, \] with
\[ a_{11} = (M + J/r^2)s^2 + (B_s + T^2/Rr^2)s + K \]
\[ a_{12} = -(J/r^2)s^2 - (T^2/Rr^2)s - K = a_{21}, \]
\[ a_{22} = (m + J/r^2)s^2 + (T^2/Rr^2)s + K + K_r. \]

There are several notable features of these equations. First of all, the equivalent damping parameter \( T^2/Rr^2 \) again appears automatically. Secondly, there is a symmetry in the second order equations, \( a_{12} = a_{21} \). This is a general feature of linear Lagrange equations that is a useful check on the correctness of the equations. A fixed point in one of the frequency responses can be found by adding the two Lagrange equations in Eq. (8). Many terms cancel with the result that

\[(Ms^2 + B_s s) z_s + (m s^2 + K_r) z_s = K z_r. \quad (9)\]

Then if \( B_s = 0 \) and \( s^2 = -K_r/m \) we find that \( z_s/z_r = -m/M \) at the frequency \( \omega = \sqrt{m/K_r} \) independent of the values of the electromechanical damper parameters. This peculiar result would be harder to derive from first order equations equivalent to the Lagrange equations.

Starting from Eq. (8) two useful transfer functions can be derived.

\[
\frac{z_s}{z_r} = \frac{[(J/r^2)s^2 + (T^2/Rr^2)s + K]K_r}{\Delta}, \quad (10)
\]
\[
\frac{z_s}{z_r} = \frac{[(M + J/r^2)s^2 + (T^2/Rr^2 + B_s)s + K]K_r}{\Delta}, \quad (11)
\]
\[
\Delta = [(M + m)J/r^2 + Mm]s^3 + [(M + m)T^2/Rr^2 + (m + J/r^2)B_s]s^2
+ (M(K + K_r) + KJ/r^2 + mK + (T^2/Rr^2)B_s)s
+ (T^2/Rr^2 + B_s)K_r + B_s K_r s + K K_r. \quad (12)
\]

These transfer functions are near the limit for hand computation but they are at least simpler to derive from two second order equations than from four first order state equations.

6. A HIGHER ORDER MODEL

In the case that the inertia of the rotary motor significantly degrades the performance one might consider the model shown in Figure 5 that mounts the rack on a spring of stiffness \( K_1 \).

![Figure 5 Compliant Element added to the Electromechanical Damper](image)

Notice now that three virtual sources are necessary to have all bonds causally augmented. This implies that the three displacement degrees of freedom are \( z_s, z_u, \) and \( \Theta \) and thus that the system will be of sixth order. The coenergies, energies and generalized efforts are readily found:

\[
T^\prime = 1/2[(M^2 + 1/2(m^2) + 1/2(J^2)],
\]

\[
V = 1/2[k(z_s - z_u)^2] + 1/2[k(z_u - z_s + r \Theta)^2],
\]

\[
E_s = -B_s z_s, \quad E_u = 0, \quad E_\Theta = -(T^2/R). \]

The Lagrange equations are fairly easy to write. In the \( \theta \)-domain they are

\[
\begin{bmatrix}
Ma^2 + B_s + K_r & -(K + k) & kr \\
-(K + k) & ms^2 + (K + K_r + k) & -kr \\
kr & -kr & J^2 + (T^2/R)s + kr^2
\end{bmatrix}
\begin{bmatrix}
z_s \\
z_u \\
\theta
\end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\]

(13)

The two corresponding transfer functions are

\[ z_s/z_r = K_1[(M^2 + B_s + K_r) (z_s + T^2/R) s + kr^2) - k r^2]/\Delta \]

\[ z_u/z_r = K_1[(M^2 + B_s + K_r) (z_u + T^2/R) s + kr^2) - k r^2]/\Delta \]

with \( \Delta \) being the determinant of the matrix in Eq.(13).

\[
\begin{bmatrix}
Ms^2 + B_s + K_r & -(K + k) & kr \\
-(K + k) & ms^2 + (K + K_r + k) & -kr \\
kr & -kr & J^2 + (T^2/R)s + kr^2
\end{bmatrix}
\]

As anyone can appreciate, at sixth order we are now approaching a limit of hand computation for all but the most patient of analysts. Computer simulation of
the bond graph is perhaps the only logical next step. However, the surprise is that adding the first two Lagrange equations again results in Eq. (9) and thus the fixed point in the frequency response discovered for the previous model remains in this extended model. This shows that Lagrange equations do have the ability to produce results that alternative equation sets would probably obscure.

7. CONCLUSIONS

Bond graph models have proven useful in the study of multi-energy-domain dynamic systems and the first order state equations are particularly useful for machine simulation. On the other hand, analytical studies may produce results that are far from obvious from observing purely numerical results. Lagrange equations may be readily derived for bond graphs and their predictable structure often allows analytical results to be obtained for simplified models that give insight into basic limitations to proposed devices.

6. REFERENCES