Optimal Structural Control under Stochastic Uncertainty: Robust Optimal Open-Loop Feedback Control

K. Marti¹ I. Stein**

¹ Federal Armed University Munich, Aerospace Engineering and Technology, 85577 Neubiberg/Munich, Germany (e-mail: kurt.marti@unibw-muenchen.de)
** Federal Armed University Munich, Aerospace Engineering and Technology, 85577 Neubiberg/Munich, Germany (e-mail: ina.stein@online.de)

Abstract: Optimal feedback controls of PD- or PID-type can be approximated very efficiently by optimal open-loop feedback controls based on optimal open-loop controls. Extending the standard construction, stochastic optimal open-loop feedback controls are constructed by using the dynamic equation (1a) in active control of dynamic structures, cf. Block (2008); Marti (2010b); Marti, K. and Stein, I. (2011); Marti (2010/2011); Nagarajaiah and Narasimhan (2007); Soong, T.T. (1988, 1990); Soong, T.T. and Costantinou, M.C. (1994); Spencer and Nagarajaiah (2003); Yang, J.N. and T.T. (1988, 1990); Soong, Marti (2010b); Marti, K. and Stein, I. (2011); Marti (2010/2011); Nagarajaiah and Narasimhan (2007); Soong, T.T. (1988, 1990); Soong, T.T. and Costantinou, M.C. (1994); Spencer and Nagarajaiah (2003); Yang, J.N. and Soong, T.T. (1988), the behavior of the control system of second order linear differential equations for the force vector $f(t, \omega,u(t))$ depending on an input or control $n$-vector function $u = u(t), t_0 \leq t \leq t_f$. Hence, the force vector $f = f(t, \omega,u(t))$ on the right hand side of the dynamic equation (1a) is given by the sum $f(t, \omega,u) = f_0(t, \omega) + f_a(t, \omega,u)$ (1b) of the applied load $f_0 = f_0(t, \omega)$ being a vector-valued stochastic process describing e.g. external loads or excitations of the structure caused by earthquakes, wind turbulence, water waves, etc., and the actuator or control force vector $f_a = f_a(t, \omega,u)$ depending on an input or control $n$-vector function $u = u(t), t_0 \leq t \leq t_f$. Here, $\omega$ denotes the random element, lying in a certain probability space $(\Omega, A, P)$, used to represent random variations. Furthermore, $M = M(t, \omega), D = D(t, \omega), K = (t, \omega)$, resp., denotes the $m \times m$ mass, damping and stiffness matrix which may depend on the time $t$ and the random element $\omega$. In many cases the actuator or control force $f_a$ is linear, i.e.

$$f_a = \Gamma_u(t, \omega)u$$ (1c)

with a certain $m \times n$ matrix $\Gamma_u = \Gamma_u(t, \omega)$. By introducing appropriate matrices, the linear system of second order differential equations (1a,b) can be represented by a system of first order differential equations as follows:

1. STOCHASTIC STRUCTURAL CONTROL SYSTEMS

1.1 Stochastic Structural Control: Active Control

In active control of dynamic structures, cf. Block (2008); Marti (2010b); Marti, K. and Stein, I. (2011); Marti (2010/2011); Nagarajaiah and Narasimhan (2007); Soong, T.T. (1988, 1990); Soong, T.T. and Costantinou, M.C. (1994); Spencer and Nagarajaiah (2003); Yang, J.N. and Soong, T.T. (1988), the behavior of the control system of second order linear differential equations for the force vector $f(t, \omega,u(t))$ depending on an input or control $n$-vector function $u = u(t), t_0 \leq t \leq t_f$. Hence, the force vector $f = f(t, \omega,u(t))$ on the right hand side of the dynamic equation (1a) is given by the sum $f(t, \omega,u) = f_0(t, \omega) + f_a(t, \omega,u)$ (1b) of the applied load $f_0 = f_0(t, \omega)$ being a vector-valued stochastic process describing e.g. external loads or excitations of the structure caused by earthquakes, wind turbulence, water waves, etc., and the actuator or control force vector $f_a = f_a(t, \omega,u)$ depending on an input or control $n$-vector function $u = u(t), t_0 \leq t \leq t_f$. Here, $\omega$ denotes the random element, lying in a certain probability space $(\Omega, A, P)$, used to represent random variations. Moreover, $M = M(t, \omega), D = D(t, \omega), K = (t, \omega)$, resp., denotes the $m \times m$ mass, damping and stiffness matrix which may depend on the time $t$ and the random element $\omega$. In many cases the actuator or control force $f_a$ is linear, i.e.

$$f_a = \Gamma_u(t, \omega)u$$ (1c)

with a certain $m \times n$ matrix $\Gamma_u = \Gamma_u(t, \omega)$. By introducing appropriate matrices, the linear system of second order differential equations (1a,b) can be represented by a system of first order differential equations as follows:
Moreover, fulfilling a certain initial condition

\[ z(t_0) = \left( \begin{array}{c} q(t_0) \\ \dot{q}(t_0) \end{array} \right) = \left( \begin{array}{c} q_0 \\ \dot{q}_0 \end{array} \right) \]  

(2f)

with given or stochastic initial values \( q_0 = q_0(\omega), \dot{q}_0 = \dot{q}_0(\omega). \)

### 1.2 Open-Loop Feedback Control

Assuming that the control force \( f_a = \Gamma_u u \) is generated by means of a PD-controller, for the input \( n \)-vector function \( u = u(t) \), we have

\[ u(t) := \varphi(t, q(t), \dot{q}(t)) \]  

(3a)

with a control law \( \varphi = \varphi(t, q, \dot{q}) \) which is determined in the following by means of the open-loop feedback method being e.g. the main tool in model predictive control, cf. Marti (2008a):

In open-loop feedback control, from an arbitrary time point \( t_b \), \( t_0 \leq t_b \leq t_f, \) to the final time point \( t_f, \) first an optimal open-loop control \( u^* = u^*(t|\{t_b, z_b\}) \) is determined on the remaining time interval \([t_b, t_f] \) and using the observed state \( z_b := z(t_b) \) at time \( t_b \). Hence, it is supposed here that at each time point \( t_b \) the state \( z_b = z(t_b) \) is available (PD-controller).

As is described in the following sections, in case of optimal control under stochastic uncertainty, by stochastic optimization methods, cf. Marti (2001, 2004, 2008b,a), a stochastic optimal open-loop control \( u^* = u^*(t|\{t_b, z_b\}) \), \( t_b \leq t \leq t_f, \) is determined for each remaining time interval \( t_b \leq t \leq t_f \) with an arbitrary starting time \( t_b \), \( t_0 \leq t_b \leq t_f \) and an initial state \( z_b \). A stochastic optimal open-loop feedback control law is defined then by

\[ \varphi(t_b, z_b) := u^*(t_b|\{t_b, z_b\}), \ t_0 \leq t_b \leq t_f. \]  

(3b)

### 2. OPEN-LOOP CONTROL PROBLEM ON THE REMAINING TIME INTERVAL \([T_B, T_F] \)

#### 2.1 Expected Total Cost Function

The performance function \( F \) for active structural control systems under stochastic uncertainty is defined, cf. Marti (2001, 2004, 2008b), by the conditional expectation of the total costs being the sum of costs \( L \) along the trajectory, arising from the displacements \( z = z(t, \omega) \) and the control input \( u = u(t, \omega) \), and possible terminal costs \( G \) arising at the final state \( z_f \).

Hence, on the remaining time interval \( t_b \leq t \leq t_f \) we have the following conditional expectation of the total cost function:

\[ F_{t_b} := \mathbb{E} \left[ \int_{t_b}^{t_f} L(t, \omega, z(t, \omega), u(t, \omega)) \, dt + G(t_f, \omega, z(t_f, \omega)) \right]_{\mathcal{F}_{t_b}}, \]

(4a)

where \( \mathcal{F}_{t_b} \) denotes the set of information up to time \( t_b \). Supposing quadratic costs along the trajectory, the costs \( L(t, \omega, z) \) along the trajectory are given by

\[ L(t, \omega, z, u) := \frac{1}{2} z^T Q(t, \omega) z + \frac{1}{2} u^T R(t, \omega) u \]

(4b)

with positive (semi) definite matrices \( Q(t, \omega), R \). The simplest case is the weight matrices \( Q, R \) are fixed. A special selection for \( Q \) reads

\[ Q = \begin{pmatrix} Q_{q} & 0 \\ 0 & Q_{\dot{q}} \end{pmatrix} \]

(4c)

with positive (semi) definite weight matrices \( Q_{q}, Q_{\dot{q}} \), resp., for \( q, \dot{q} \). Furthermore, \( G = G(t_f, \omega, z(t_f, \omega)) \) describes possible terminal costs. In case of endpoint control \( G \) is defined by

\[ G(t_f, \omega, z(t_f, \omega)) = \frac{1}{2} (z(t_f, \omega) - z_f(\omega))^T G_f(\omega) (z(t_f, \omega) - z_f(\omega)), \]

(4d)

where \( G_f = G_f(\omega) \) is a positive (semi)definite, possible random weight matrix, and \( z_f = z_f(\omega) \) denotes the (possible probabilistic) final state.

**Remark 2.1.** Instead of \( \frac{1}{2} u^T R(t, \omega) u \), in the following we consider also more general control cost function \( C \) a convex function in the control \( u \).

#### 2.2 Optimal Open-Loop Control Problem under Stochastic Uncertainty

Having the differential equation with random coefficients describing the behavior of the dynamic mechanical structure under uncertainty and the costs arising from displacements along the trajectory and at the terminal state, on a given remaining time interval \([t_b, t_f] \) an optimal open-loop control \( u^* = u^*(t|\{t_b, z_b\}), t_b \leq t \leq t_f, \) is a solution
of the following optimal control problem under stochastic uncertainty:

\[
\min_{u} \mathbb{E} \left[ \int_{t_0}^{t_f} L(t, \omega, z(t), u(t)) \, dt + G(t_f, \omega, z(t_f)) \right] \quad (5a)
\]

s.t. \( \dot{z}(t, \omega) = A(t, \omega)z(t, \omega) + B(t, \omega)u(t) + b(t, \omega), \)

\( a.s., \quad t_0 \leq t \leq t_f \)

\( z(t_0, \omega) = z_0 \) (given)

\( u(t) \in U_t, \quad t_0 \leq t \leq t_f \),

where \( U_t, t_0 \leq t \leq t_f \), denotes the convex set of feasible controls at time point \( t \). An important property of (5a-d) is stated next:

**Lemma 1.** Suppose that the feasible domain \( U_t \) is convex for each time point \( t, t_0 \leq t \leq t_f \). If the terminal cost function \( G = G(t_f, \omega, z) \) is convex in \( z \), then the optimal control problem under stochastic uncertainty (5a-d) is a convex optimization problem.

### 2.3 The Stochastic Hamiltonian of (5a-d)

According to Marti (2010a, 2008a), the stochastic Hamiltonian \( \mathcal{H} \) related to the optimal control problem under stochastic uncertainty (5a-d) reads:

\( \mathcal{H}(t, \omega, z, u) := L(t, \omega, z, u) + g^T(t, \omega, z, u) \)

\( = \frac{1}{2} z^T Q(t, \omega)z + C(t, \omega, u) \)

\( + g^T(A(t, \omega)z + B(t, \omega)u + b(t, \omega)) \). \hspace{1cm} (6a)

### 2.4 Expected Hamiltonian (with respect to the time interval \([t_0, t_f]\) and information \( \mathcal{A}_{t_0} \))

For the definition of a \( H \)-minimal control the conditional expectation of the stochastic Hamiltonian is needed:

\( \mathbb{E}^{(b)} := \mathbb{E}(\mathcal{H}(t, \omega, z, u)|\mathcal{A}_{t_0}) \)

\( = \mathbb{E} \left[ \frac{1}{2} z^T Q(t, \omega)z + C(t, \omega, u) \right] |\mathcal{A}_{t_0} \)

\( + \mathbb{E}(g^T B(t, \omega)u)|\mathcal{A}_{t_0} \)

\( = \mathbb{E}(B(t, \omega)^T y(t, \omega)|\mathcal{A}_{t_0})^T u + \ldots \)

\( = \mathbb{E}(C(t, \omega, u)|\mathcal{A}_{t_0}), \quad u \in \mathcal{U}^n. \) \hspace{1cm} (6b)

\( h(t) := B(t, \omega)^T y(t) \)

\( = \mathbb{E}(B(t, \omega)^T y(t, \omega)|\mathcal{A}_{t_0}) = h(t, t_0, \mathcal{A}_{t_0}), \quad t \geq t_0, \) \hspace{1cm} (6c)

\( \mathbb{C}^{b}(t, u) := \mathbb{E}(C(t, \omega, u)|\mathcal{A}_{t_0}), \quad u \in \mathcal{U}^n. \) \hspace{1cm} (6d)

### 2.5 H-Minimal Control on \([t_0, t_f]\)

In order to formulate the two-point boundary value problem for a stochastic optimal open-loop control \( u^* = u^*(t|[t_0, z_0]), t_0 \leq t \leq t_f, \) we need first a \( H \)-minimal control

\( \hat{u}^* = \hat{u}^*(t, z(t, \cdot), y(t, \cdot)), t_0 \leq t \leq t_f, \) defined, cf. Marti (2008a), for \( t_0 \leq t \leq t_f \) as a solution of the following convex finite dimensional stochastic optimization problem, cf. Marti (2008b):

\[
\min_{u} \mathbb{E}(H(t, \omega, z(t, \omega), y(t, \omega), u)|\mathcal{A}_{t_0}) \quad (7a)
\]

s.t. \( u \in U_t, \)

where \( z = z(t, \omega), y = y(t, \omega) \) are certain (state, adjoint state, resp.) trajectories.

According to (6a-d), (7a,b), in the present case, for the \( H \)-minimal control we have

\( \hat{u}^* = \hat{u}^*(t, z(t, \cdot), y(t, \cdot)) = v(t, h(t)), \) \hspace{1cm} (8a)

where

\[
v(t, h(t)) := \arg\min_{u \in U_t} \left( \mathbb{C}^b(t, u) + h(t)^T u \right) \quad \text{for} \ t \geq t_0. \hspace{1cm} (8b)
\]

Based on the assumption in Remark 2.1, the expected cost function \( \mathbb{C}^b = \mathbb{C}^b(t, u) \) is convex in the control \( u, \) cf. Marti (2008b). An important example is the expected control cost function \( \mathbb{C}^b(t, u) = \frac{1}{2} u^T \mathcal{R}(t) u \) with a positive semidefinite matrix \( \mathcal{R}(t) := \mathbb{E}(R(t, \omega)|\mathcal{A}_{t_0}). \) Hence, the necessary and sufficient conditions for \( \hat{u}^* \) can be formulated by means of the Kuhn-Tucker conditions, see e.g. Marti (2008b), of problem (8b). Especially, if there are no control constraints, hence, if \( U_t = \mathbb{R}^n \), then the necessary and sufficient condition for \( \hat{u}^* \) reads

\[
\nabla \mathbb{C}^b(t, u) + h(t) = 0. \hspace{1cm} (9a)
\]

If the gradient operator \( u \mapsto \mathbb{C}^b(t, u) \) is a 1-1-operator, then the solution of (9a) reads

\[
u = v(t, h(t)) = \nabla \mathbb{C}^b(t, -1)(-h(t)). \hspace{1cm} (9b)
\]

Hence, if \( U_t = \mathbb{R}^n \), then with (6c,d) and (8a,b) we have

\( \hat{u}^*(t, h(t)) = v(t, h(t)) = \nabla_u \mathbb{C}^b(t, -1)(-B(t)^T y(t)) \).

### Remark 2.2.

Note that in the present case the \( H \)-minimal control \( \hat{u}^* \) depends on the dual or adjoint trajectory \( y = y(t, \cdot) \) only.

### 3. CANONICAL (HAMILTONIAN) SYSTEM

Suppose that there are no control constraints. Thus, \( U_t = \mathbb{R}^n. \)

In the following we assume that a \( H \)-minimal control \( \hat{u}^* = \hat{u}^*(t, z(t, \cdot), y(t, \cdot)), t_0 \leq t \leq t_f, \) i.e. a solution of the stochastic optimization problem (7a,b) of the type (9c) is available. Hence, \( \hat{u}^* = \hat{u}^*(t, h(t)) = v(t, h(t)). \)
According to Marti (2008a, 2010a), a stochastic optimal open-loop control $u^* = u^*(t|t_0, z_b), t_0 \leq t \leq t_f$, hence, a solution of the stochastic optimal control problem (5a-d), can be represented then by

$$u^*(t|t_0, z_b) = \tilde{u}^*(t, z^*(t, \cdot), y^*(t, \cdot)) = \tilde{u}^*(t, y^*(t, \cdot)), \quad t_0 \leq t \leq t_f,$$

where the trajectories $z^* = z^*(t, \omega), y^* = y^*(t, \omega), t_0 \leq t \leq t_f$ are solutions of the following stochastic two-point boundary value problem related to (5a-d):

**Theorem 2.** If $z^* = z^*(t, \omega), y^* = y^*(t, \omega), t_0 \leq t \leq t_f$, is a solution of

$$\dot{z}(t, \omega) = A(t, \omega)z(t, \omega) + B(t, \omega)\nabla_u C(t, \cdot)^{-1}\left(-B(t)^T y(t)\right) + b(t, \omega), \quad t_0 \leq t \leq t_f,$$

$$y(t, \omega) = \nabla_z G(t, \omega, z(t, \omega)), \quad t_0 \leq t \leq t_f,$$

then the function $u^* = u^*(t|t_0, z_b), t_0 \leq t \leq t_f$, defined by (10) is a stochastic optimal open-loop control for the remaining time interval $t_0 \leq t \leq t_f$.

### 3.1 Basic Solution Procedure for (11a-d)

Obviously, (11a-d) is a coupled system of an initial and a terminal value problem for the random (primal) trajectory $z = z(t, \omega)$ and its random adjoint trajectory $y = y(t, \omega)$. Furthermore, the subsystem (11c,d) is linear in the unknowns $(z(t, \omega), y(t, \omega))$, and subsystem (11a,b) can be interpreted, for given adjoint trajectory $y = y(t, \omega)$, as an inhomogeneous linear system for the primal trajectory $z = z(t, \omega)$. Hence, for given adjoint, primal trajectory $y = y(t, \omega), z = z(t, \omega)$, resp., we have an inhomogeneous linear initial, terminal value problem for the trajectory $z = z(t, \omega), y = y(t, \omega)$, respectively. Thus, having the corresponding fundamental matrix for the homogeneous system of ordinary random differential equation related to (11a), (11e), resp., the subsystems (11a,b), (11c,d), depending on the system matrix $A = A(t, \omega)$, $-A^T = -A^T(t, \omega)$, resp., can be solved explicitly for given trajectory $y = y(t, \omega), z = z(t, \omega)$, respectively. The solution can then be obtained by the procedure described in the following:

**Lemma 3.** The basic solution procedure consists of the following steps:

**Step 1a** For given adjoint trajectory $y = y(t, \omega)$ solve the inhomogeneous linear initial value problem (11a,b) for the primal trajectory $z = z(t, \omega);

**Step 1b** For given primal trajectory $z = z(t, \omega)$ solve the inhomogeneous linear terminal value problem (11c,d) for the adjoint trajectory $y = y(t, \omega);

**Step 2** Insert for given adjoint trajectory $y = y(t, \omega)$, the solution of (11a,b) into the solution of (11c,d);

**Step 3** Solve (e.g. iteratively) the resulting fixed point problem for the adjoint variable $y = y(t, \omega);

**Step 4** Insert the adjoint variable $y = y(t, \omega)$ into equation (9a) to obtain - together with equation (10) - a stochastic optimal open-loop control $u^* = u^*(t|t_0, z_b)$ for the remaining time interval $t_0 \leq t \leq t_f$;

**Step 5** Define the stochastic optimal open-loop feedback control law by means of equation (3b).

**Steps 1a and 1b** Let $y = y(t, \omega)$ denote a given adjoint trajectory. According to (11a,b), for the primal trajectory $z = z(t, \omega)$ we have the following Initial Value Problem (IVP):

$$\dot{z}(t, \omega) = A(t, \omega)z(t, \omega) + B(t, \omega)\nabla_u C(t, \cdot)^{-1}\left(-B(t)^T y(t)\right) + b(t, \omega), \quad t_0 \leq t \leq t_f,$$

$$y(t, \omega) = \nabla_z G(t, \omega, z(t, \omega)), \quad t_0 \leq t \leq t_f,$$

where

$$\beta(s, \omega) = B(s, \omega)\nabla_u C(s, \cdot)^{-1}\left(-B(s)^T y(s)\right) + b(s, \omega).$$

If the expected control costs are given by $\nabla_u C(t, u) = \frac{1}{2} u^T \overline{R}^b(t) u$, with a positive definite matrix $\overline{R}^b(t) := \mathbb{E} (R(t, \omega) | \mathcal{A}_t)$, then

$$\beta(s, \omega) = -\nabla_z G(t, \omega, z(t, \omega)).$$

Denoting by $\Psi = \Psi(t, \omega)$ the fundamental matrix related to the system matrix $A = A(t, \omega)$, the adjoint system (11c,d) has the explicit solution:

$$y(t, \omega) = \Psi(t, \omega) \left(1_t \Psi(s, \omega)^{-1} Q z(s, \omega) ds \right), \quad t_0 \leq t \leq t_f.$$

Hence, in order to carry out Steps 1a, 1b, the fundamental matrix related to the system matrix $A = A(t, \omega)z(t, \omega), -A^T = -A^T(t, \omega)$, resp., is needed. According to the dependencies of the system matrices on time $t$ and the random element $\omega$, exact and approximative fundamental matrices may be constructed. However, first of all, the following relationship between the fundamental matrices of the primal and adjoint system can be shown:

**Lemma 4.** Let $A = A(t, \omega), \Psi = \Psi(t, \omega)$ denote the fundamental matrix with respect to the problems (11a,b), (11c,d), respectively. Then, $\Psi(t, \omega) = (\Lambda(t, \omega)^{-1})^T$.
4. APPROXIMATIONS OF THE FUNDAMENTAL MATRICES

Approximates $\tilde{A} = \tilde{A}(t, \omega)$, $\tilde{\Psi} = \tilde{\Psi}(t, \omega)$, of the fundamental matrix $\Lambda = \Lambda(t, \omega)$, $\Psi = \Psi(t, \omega)$, resp., can be obtained by considering suitable approximates $\tilde{A} = \tilde{A}(t, \omega)$, $\tilde{\Lambda} = \tilde{\Lambda}^T = -\tilde{A}^T(t, \omega)$ of the system matrices $A(t, \omega), -A^T(t, \omega)$ of the initial, terminal value problems (11a,b), (11c,d), respectively.

4.1 Approximation by a Constant System Matrix

As was shown in $[5 - 8]$, the solution process of the 2-point value problem (11a-d) can be continued rather easily in case of a constant system matrix $A$, i.e., $A$ does not depend on time $t$ and the random element $\omega$. In the present open-loop feedback approach, see Section (1.2), stochastic optimal open-loop controls are determined on each remaining time interval $[t_i, t_f]$, but evaluated only at the initial points $t_i, t_f \leq t \leq t_f$. Thus, in case of a moderate change of the system matrix $A = A(t, \omega)$ along the interval $[t_i, t_f]$, the system matrix can be approximated on $[t_i, t_f]$ by

$$\tilde{\Lambda}_{t_i} := E \left( A(t_i, \omega) | \mathcal{F}_{t_i} \right).$$

Using then the matrix exponential function $e^{At}, t \in \mathcal{R}$, the fundamental matrix $\Lambda(t, \omega)$ can be approximated as follows:

$$\Lambda(t, \omega) \approx e^{\tilde{\Lambda}_{t_i} (t-t_i)}, \quad t_i \leq t \leq t_f. \quad (14b)$$

Furthermore, according to Lemma 4, the related fundamental matrix $\Psi(t, \omega)$ can be approximated by

$$\Psi(t, \omega) \approx e^{-\tilde{\Lambda}^T_{t_i} (t-t_i)}, \quad t_i \leq t \leq t_f. \quad (14c)$$

With these approximations of the fundamental matrices, the stochastic optimal open-loop feedback controls can be determined as shown in Marti (2010b); Marti, K. and Stein, I. (2011); Marti (2010/2011).

4.2 Approximation by a Piecewise Constant System Matrix

The approximation method described in the above Section (4.1) can be extended by assuming that the interval $[t_i, t_f]$ is partitioned into several disjoint subintervals $T_0 := (-\infty, 0], T_i := (t_{i-1}, t_i], i = 1, ..., r$, with $t_0 := t_i$ and $t_r := t_f$. Moreover, for the extension of the constant approximation (14a), we consider the events $\mathcal{A}_k, k = 0, 1, ..., r-1$, i.e., the set of elementary events $\omega$ occurring during the time-intervals $T_i, i = 0, 1, ..., r-1$. The system matrix $A = A(t, \omega)$ is then approximated as follows:

$$A(t, \omega) \approx \tilde{\Lambda}_{t_i, t_k} := E \left( A_{t_i, t_k}(\omega) | \mathcal{F}_{t_i} \right),$$

$$t \in T_i, i = 1, ..., r, \omega \in \mathcal{A}_k, k = 0, 1, ..., r-1. \quad (15)$$

Based on the interval partition $t_0 < t_1 < \cdots < t_{r-1} < t_r = t_f$, the events $\mathcal{A}_k, k = 0, 1, ..., r-1$ and using again the matrix exponential function $e^{At}, t \in \mathcal{R}$, for each elementary event lying in an event $\mathcal{A}_k, k = 0, 1, ..., r-1$, a continuous, piecewise differentiable approximation

$$\Lambda(t, \omega) \approx \tilde{\Psi}(t, \omega) \quad (16a)$$

of the fundamental matrix $\Lambda(t, \omega)$ can be obtained as a solution of the following matrix initial value problem

$$\dot{\tilde{X}}(t, \omega) = \tilde{A}(t, \omega)\tilde{X}(t, \omega), \quad \tilde{X}(t_b, \omega) = I \text{ (unit matrix)} \quad (16b)$$

with

$$\tilde{A}(t, \omega) = \begin{cases} \tilde{\Lambda}_{t_i, t_k} & \text{for } t_k \leq t < t_i \\ \tilde{\Lambda}_{t_i, t_k} & \text{for } t_1 \leq t < t_2 \\ \vdots & \text{for } t_{(r-1)} \leq t \leq t_f. \end{cases} \quad (16c)$$

In order to get a continuous, piecewise differential solution of the above matrix initial value problem, appropriate initial values $\tilde{X}(t_i, \omega) = X_i(\omega)$, $1 \leq i \leq r-1$ for the remaining $r - 1$ subintervals $[t_{(i-1)}, t_i], i = 2, ..., r$ are needed. These matrices can be given recursively by

$$X_i(\omega) := e^{(t_i-t_{(i-1)})\tilde{\Lambda}_{t_{(i-1)}, t_i}}X_{(i-1)}(\omega), \quad 1 \leq i \leq r-1, \quad \omega \in \mathcal{A}_k,$$

$$X_0(\omega) := I. \quad (16d)$$

Lemma 5. Under the above assumptions, for $\omega \in \mathcal{A}_k$ the continuous, piecewise differentiable solution of (16b) reads:

$$\tilde{X}(t, \omega) = \begin{cases} e^{(t-t_0)\tilde{\Lambda}_{t_0, t_k}} & \text{for } t_k \leq t < t_0 \\ e^{(t-t_k)\tilde{\Lambda}_{t_k, t_{(k+1)}}} \prod_{j=1}^{r-1} e^{(t_{(r-k-1)})\tilde{\Lambda}_{t_{(r-k-1)}, t_{(r-k)}})} & \text{for } t_{(r-k)} \leq t \leq t_{(r-k-1)}. \end{cases} \quad (17)$$

5. CONCLUSION

Using stochastic optimization methods, deterministic substitute problems for optimal control problems under stochastic uncertainty have been modeled by minimizing the expected total costs arising from the costs along the trajectory caused by displacements from the nominal trajectory and from the corresponding control corrections. In many practical cases the resulting control problem has a linear-quadratic structure, i.e. a linear plant differential equation and quadratic cost functions.

Due to the great theoretical and practical advantages of open-loop feedback controls, e.g. in applications to Model Predictive Control, stochastic optimal open-loop feedback controls have been introduced by taking into account the available information about the random parameter variations in the control problem. Corresponding to the standard case, stochastic optimal open-loop feedback controls are based on the family of stochastic optimal open-loop controls on the remaining time intervals: For finding stochastic optimal open-loop controls on the remaining time intervals $t_b \leq t \leq t_f$ with $t_0 \leq t_b \leq t_f$, the stochastic Hamiltonian function of the control problem has been introduced. Then, the class of $H-$ minimal controls can be determined by solving a finite-dimensional stochastic optimization problem for minimizing the conditional expectation of the stochastic Hamiltonian subject to the remaining deterministic control constraints at each time
Having a $H^-$ minimal control, the related two-point boundary value problem with random parameters can be formulated for the computation of the stochastic optimal state- and adjoint state trajectory. Inserting then the resulting trajectories into the $H^-$ minimal control, stochastic optimal open-loop controls are found on arbitrary remaining time intervals. Evaluating then these controls on the corresponding initial time points $t_b$, $t_0 \leq t_b \leq t_f$, only, a stochastic optimal open-loop feedback control law follows.

In order to transfer the semi-analytical solution method for two-point boundary value problems, developed for constant system matrices also to time-dependent random ones, several approximation methods for computing the fundamental matrix have been introduced, and corresponding error estimates were given.

REFERENCES


